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Space Guidance Analysis Memo #49-64

TO: SGA Distribution
FROM: Edward M. Copps, Jr.
DATE: December 3, 1964
SUBJECT: Noise Through a Derived Rate Controller

I. Summary

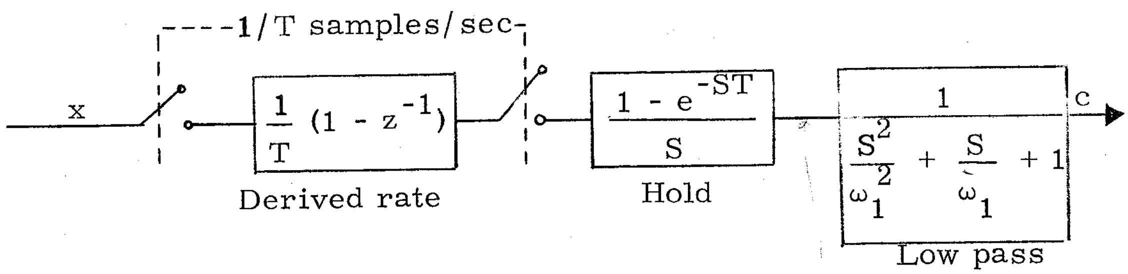
Noise, characterized by an exponential auto correlation function is sampled, and differenced to form an approximation to the derivative. The output of the difference equation is passed through a zero order hold and a 2nd order "low pass" cut-off filter. A series of graphs are obtained with relevant parameters rendered non-dimensional.

The graphs illustrate one aspect of the problem of using derived rate: namely, the noise on the output can be considerably larger than with a continuous system. The results can be used to choose sampling rates and to judge effects of sampling on noise transmission.

II. The System

The system is illustrated in Fig. 1. The input signal is a random variable x(t), characterized by an auto correlation function which we choose simply:

$$\phi_{xx}(\tau) = e^{-|\omega_n \tau|} \tag{1}$$



The System
Fig. 1

III. Parameters

The following parameters are of interest:

a) T, the sample period in seconds

b) ω_n the cut-off frequency of the noise spectrum

c) ω_1 the cut-off of the low pass filter

We immediately relate the noise cut-off to the sampling interval by

$$\omega_n = N/T \tag{2}$$

IV. Analysis

The z transform of the power density spectrum of the sampled input signal is

$$S_{xx}^*(z) = \frac{1}{T} \frac{1 - e^{-2N}}{(1 - e^{-N}z^{-1})(1 - e^{-N}z)} \tag{3}$$

To pass noise through a difference equation

$$S_{xx}^*(z) = S_{xx}^*(z)\, c(z)\, c(z^{-1}) \tag{4}$$

where

$$c(z)\, c(z^{-1}) = \frac{1}{T^2} (2 - (z + z^{-1})) \tag{5}$$

Performing this operation and then changing from the z transform to the Fourier transform, the power spectrum of the sampled output of the difference equation is

$$S_{yy}^*(j\omega) = \frac{2}{T^3} \frac{(1 - e^{-2N})(1 - \cos T\omega)}{(1 + e^{-2N} - 2e^{-N} \cos T\omega)} \tag{6}$$

To pass this through the continuous system, the following formula applies

$$S_{cc}(j\omega) = S_{yy}^*(j\omega)\, H(j\omega)\, H(-j\omega) \tag{7}$$

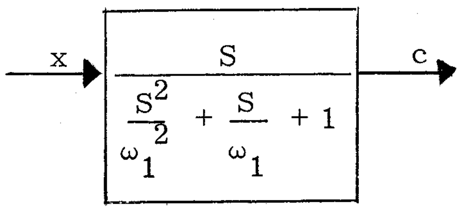
where

$$H(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega\left(\frac{(j\omega)^2}{\omega_1^2} + \frac{j\omega}{\omega_1} + 1\right)} \tag{8}$$

Yielding finally

$$S_{cc}(j\omega) = \frac{4}{T^3} \frac{(1 - e^{-2N})(1 - \cos T\omega)^2}{\omega^2\left(1 - \frac{\omega^2}{\omega_1^2} + \frac{\omega^4}{\omega_1^4}\right)\left(1 + e^{-2N} - 2e^{-N}\cos T\omega\right)} \tag{9}$$

This equation has been checked to assure that it is equivalent to the formula for a continuous system of Fig. 2 as sampling frequency is increased without limit.



Equivalent Continuous System

Fig. 2

V. Numerical Evaluation

The following steps are done to non-dimensionalize the solution.

1) relate sampling time to the frequency of the low pass filter by

$$T = \frac{2\pi}{\omega_1 f}$$

Now f is the number of samples per cycle.

2) relate the auto correlation time to ω_1 by

$$\omega_n = N/T = \ell\omega_1 = \frac{2\pi\ell}{Tf}$$

Now ℓ is the ratio of the frequency of noise cut-off to the frequency of the low pass filter cut-off.

3) Plot the function $S_{cc}(j\omega)/\omega_1$ versus M, where

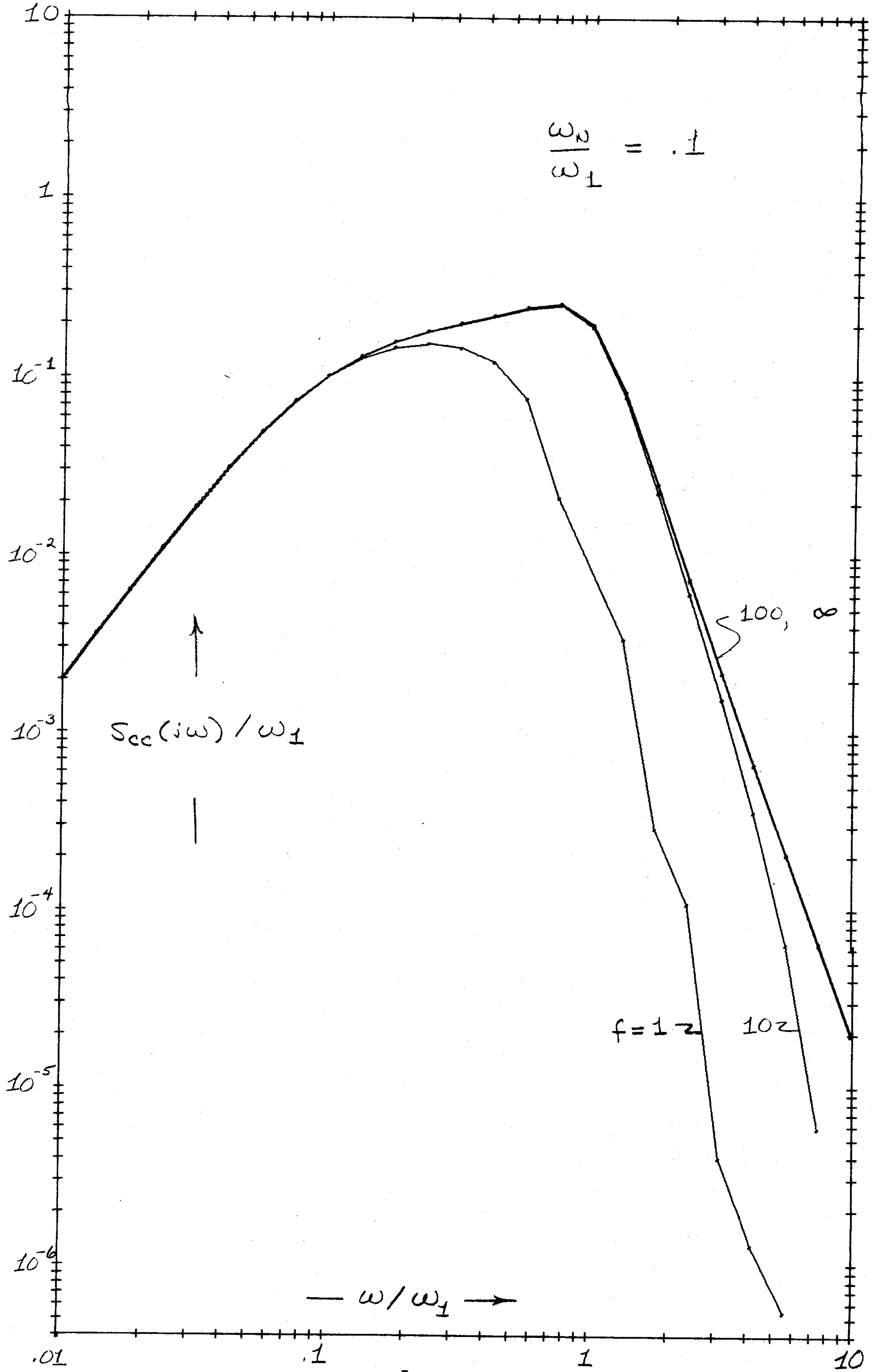
$$M = \omega/\omega_1$$

4) Note that the rms value associated with the input signal is always unity

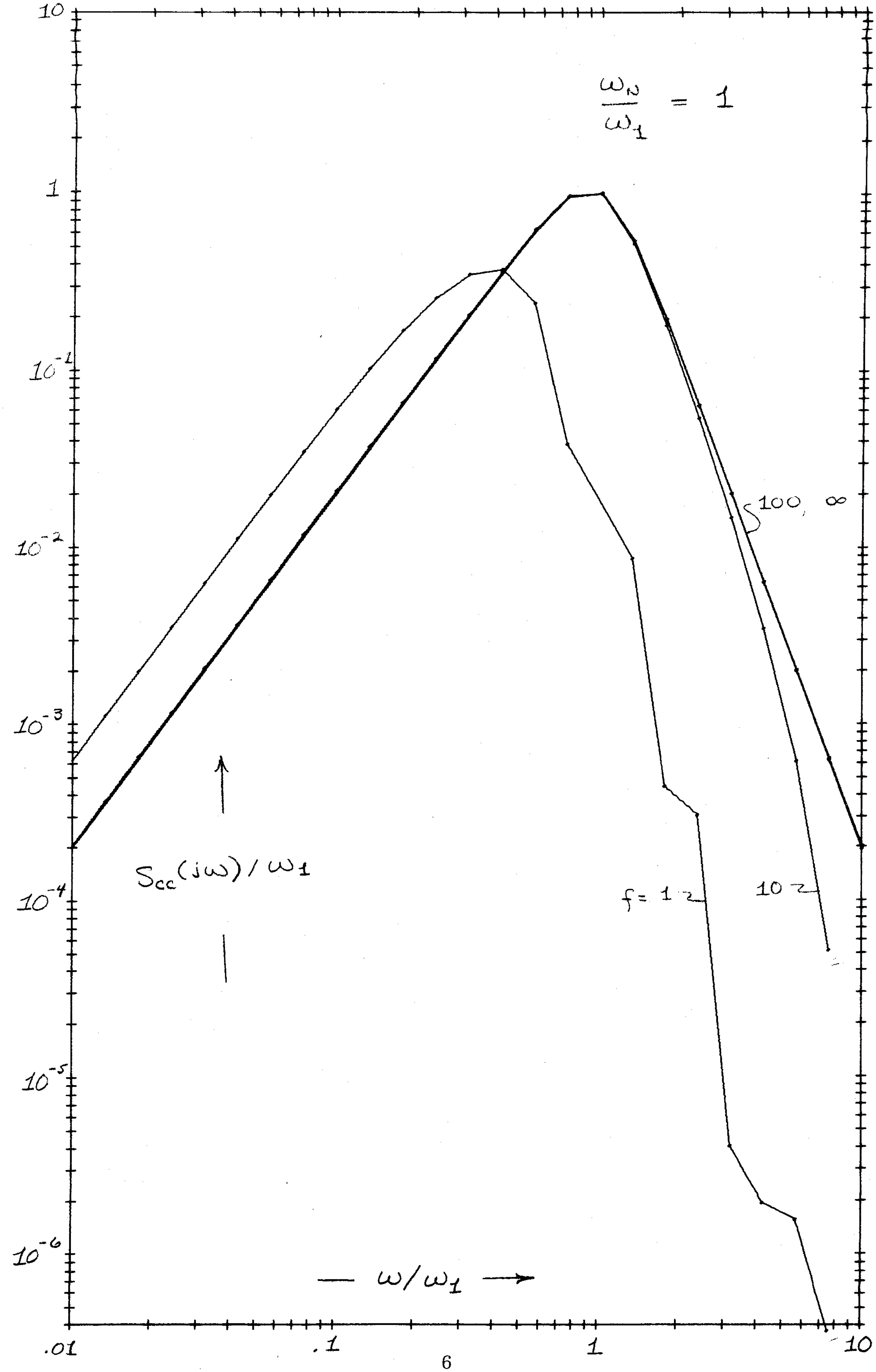
$$\phi_{xx}(0) = 1$$

The formula, in terms of the above variables, becomes

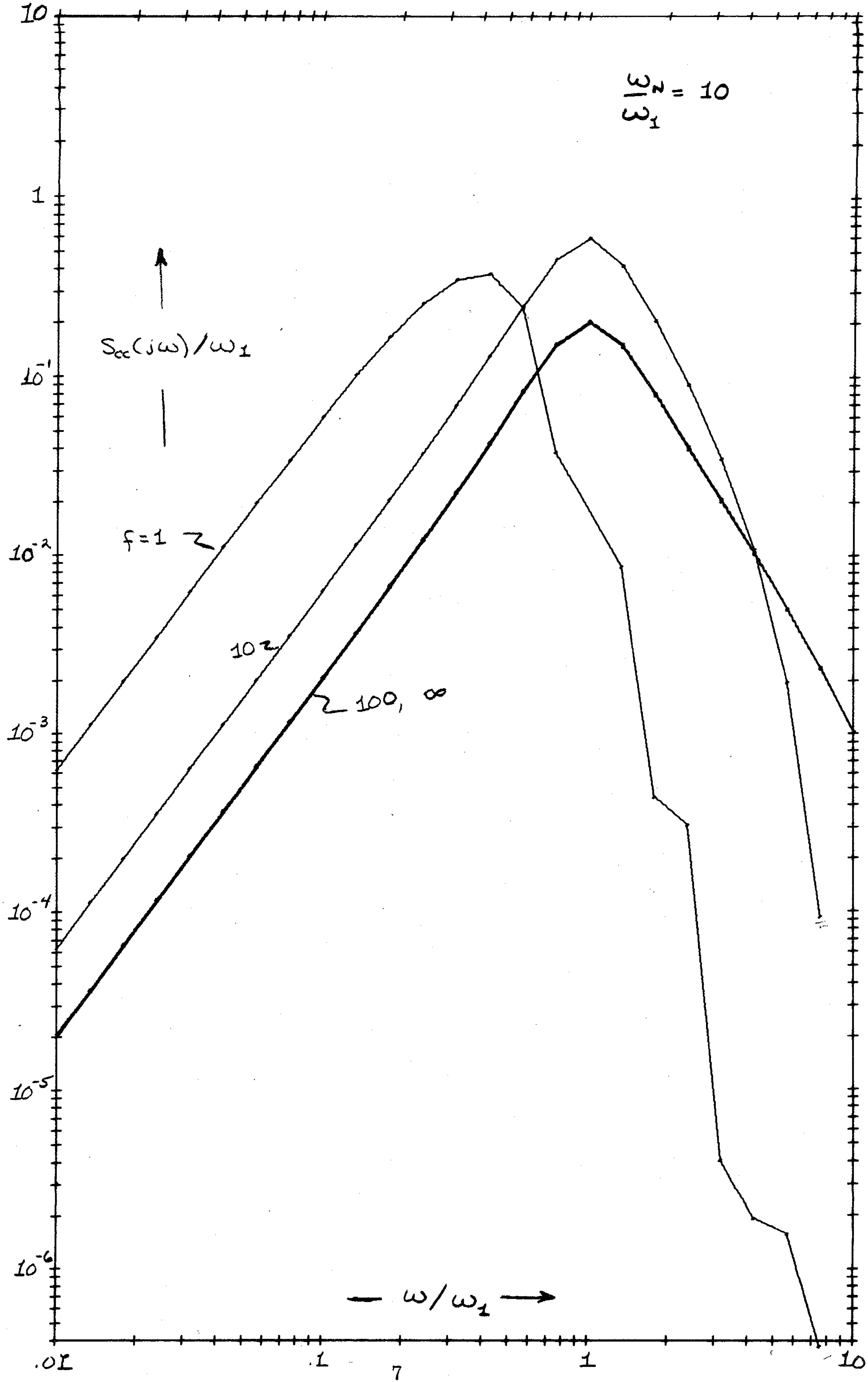
$$S_{cc}(j\omega) = \frac{\omega_1 f^3 (1 - e^{-4\pi\ell/f})(1 - \cos(2\pi M/f))^2}{2\pi^3 M^2 (1 - M^2 + M^4)(1 + e^{-4\pi\ell/f} - 2e^{-2\pi\ell/f}\cos(2\pi M/f))} \tag{10}$$



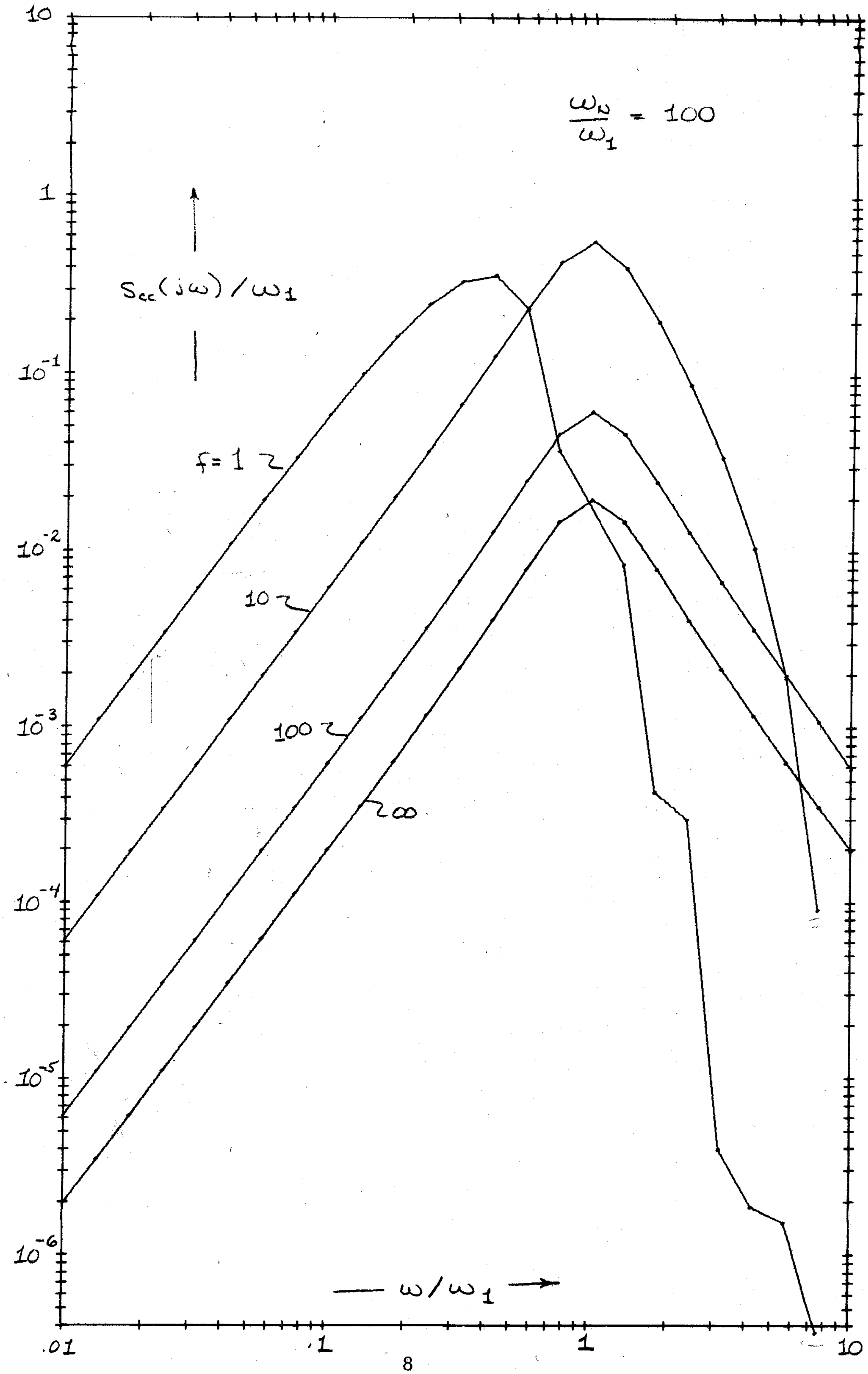
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\frac{\omega_N}{\omega_1} = .1$
$S_{cc}(i\omega) / \omega_1$
100, ∞
f = 1 z
10 z
ω / ω_1
.01
.1
1
10



10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\frac{\omega_N}{\omega_1} = 1$
100, ∞
$S_{cc}(j\omega)/\omega_1$
f = 1
10
ω/ω_1
.01
.1
1
10



10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\frac{\omega_N}{\omega_1} = 10$
$S_{cc}(j\omega)/\omega_1$
f=1
10
100, ∞
ω/ω_1
.01
.1
1
10


$\frac{\omega_N}{\omega_1} = 100$
$S_{cc}(j\omega)/\omega_1$
f = 1
10
100
∞
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
ω/ω_1
.01
.1
1
10