Massachusetts Institute of Technology
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Space Guidance Analysis Memo # 48-64

TO: SGA Distribution

FROM: William Marscher

DATE: November 1, 1964

SUBJECT: Collection of Conic Equations

The purpose of this memo is to collect in one place a group of equations from SGA Memo # 23-64 Rev. 1 which are useful in computing conic sections. The definition of variables is found at the end of this memo.

I. Relating the true and eccentric anomaly differences and their parabolic and hyperbolic equivalents:

(1) $\cot\left(\frac{\theta}{2}\right) = \frac{r_0}{\sqrt{pa}} \cot\left(\frac{\Delta E}{2}\right) + \cot(\gamma_0)$ (ellipse)

(2) $\cot\left(\frac{\theta}{2}\right) = \frac{2 r_0}{\sqrt{p}\ x} + \cot(\gamma_0)$ (parabola)

(3) $\cot\left(\frac{\theta}{2}\right) = \frac{r_0}{\sqrt{-pa}} \coth\left(\frac{\Delta G}{2}\right) + \cot(\gamma_0)$ (hyperbola)

(4) $\cot\left(\frac{\theta}{2}\right) = \frac{r_0\left[1 - \alpha x^2 S(\alpha x^2)\right]}{\sqrt{p}\ x\ C(\alpha x^2)} + \cot(\gamma_0)$ (universal)

II. Relating time of flight to conic parameters and the eccentric anomaly differences (or equivalent):

(5) $\sqrt{\mu\alpha^3}\ t = \Delta E + \sqrt{p\alpha}\ \cot(\gamma_0)\left|1 - \cos(\Delta E)\right| - (1 - r_0\alpha)\sin(\Delta E)$ (ellipse)

(6) $\sqrt{\mu}\ t = \frac{1}{2}\sqrt{p}\ \cot(\gamma_0)x^2 + \frac{1}{6}x^3 + r_0 x$ (parabola)

(7) $\sqrt{-\mu\alpha^3}\ t = -\Delta G + \sqrt{-p\alpha}\ \cot(\gamma_0)\left|\cosh(\Delta G) - 1\right| + (1 - r_0\alpha)\sinh(\Delta G)$ (hyperbola)

(8) $\sqrt{\mu}\ t = \sqrt{p}\ \cot(\gamma_0)x^2\ C(\alpha x^2) + (1 - r_0\alpha)x^3 S(\alpha x^2) + r_0 x$ (universal)

III. Relating conic parameters and the true anomaly difference (all universal):

(9) $$\frac{p}{r_0} = \frac{1 - \cos(\theta)}{\frac{r_0}{r_1} - \cos(\theta) + \sin(\theta)\cot(\gamma_0)}$$ (polar equation)

(10) $$\frac{r_0}{\mu} v_0^2 = 2 - r_0\alpha$$ (energy integral)

(11) $$r_0\alpha = 2 - \frac{p}{r_0}\left[1 + \cot^2(\gamma_0)\right]$$

(12) $$\cot\left(\frac{\theta}{2}\right) = \frac{\cot(\gamma_0) + \frac{r_0}{r_1}\cot(\gamma_1)}{\left(1 - \frac{r_0}{r_1}\right)}$$ $r_0 \neq r_1$

(13) $$\frac{p}{r_0} = \frac{2\left(\frac{r_0}{r_1} - 1\right)}{\left(\frac{r_0}{r_1}\right)^2\left[1 + \cot^2(\gamma_1)\right] - \left[1 + \cot^2(\gamma_0)\right]}$$

(14) $\sqrt{p\mu} \cot(\gamma) = \bar{r} \cdot \bar{v}$

(15) $$v_{circum} = \frac{\sqrt{p\mu}}{r}$$

(16) $$v_{radial} = \frac{\sqrt{p\mu}}{r} \cot(\gamma)$$

IV. Useful identity:

(17) $$\cos(\phi) = \left[\frac{\cot^2\left(\frac{\phi}{2}\right) - 1}{\cot^2\left(\frac{\phi}{2}\right) + 1}\right]$$

$$\sin(\phi) = \text{sign}\left(\cot\left(\frac{\phi}{2}\right)\right)\sqrt{1 - \cos^2(\phi)}$$

V. Allowable Range of $\cot(\gamma_0)$ for Lambert Constraints (r_0, r_1, θ) and Reentry Constraints $(r_0, r_1, \cot(\gamma_0))$.

(a) Lambert $(\theta < 180^\circ)$

$$c < \cot(\gamma_0) < (b+)$$

where

$$b = \cot\left(\frac{\theta}{2}\right) \pm \sqrt{\frac{2\,\frac{r_0}{r_1}}{1 - \cos(\theta)}}$$

The sign after b indicates the sign of the radical

$$c = \frac{\cos(\theta) - \frac{r_0}{r_1}}{\sin(\theta)}$$

for $\cot(\gamma_0) = b+$, $t = \infty$, $e = 1$, a flight through infinity on a parabola.

for $\cot(\gamma_0) = c$, $t = 0$, $e = \infty$, a direct straight line flight from r_0 to r_1.

For the values of $\cot(\gamma_0)$ from b+ to c, the conic goes from the elliptic to the hyperbolic region. These two regions are divided by the parabola for which $\cot(\gamma_0) = b-$.

(b) Lambert $(\theta > 180^\circ)$

The same as the $\theta < 180^\circ$ case except that $c = -\infty$

for $\cot(\gamma_0) = -\infty$, $t = 0$, $e > 1$, a flight down r_0 to the focus and up r_1.

(c) <u>Reentry $\left(\frac{r_0}{r_1} > 1\right)$</u>

$$g < \cot(\gamma_0) < d$$

where
$$d = \sqrt{\left[1 + \cot^2(\gamma_0)\right] \frac{r_0}{r_1} - 1}$$

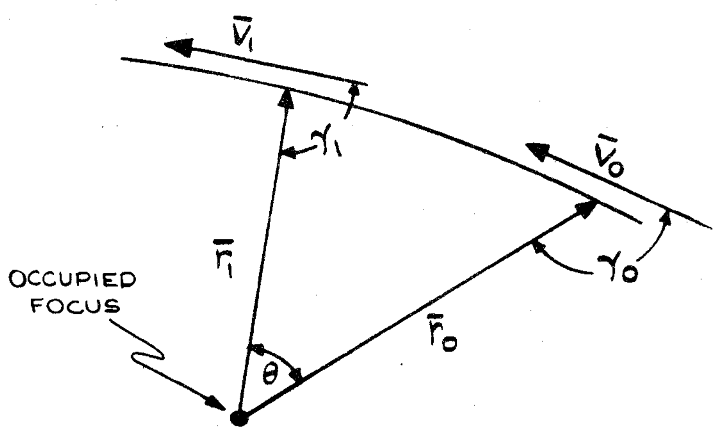
$$g = -\sqrt{\left[1 + \cot^2(\gamma_0)\right] \left(\frac{r_0}{r_1}\right)^2 - 1}$$

for $\cot(\gamma_0) = d$, $t = \infty$, $e = 1$, a flight through infinity on a parabola.

for $\cot(\gamma_0) = g$, $t = 0$, $e = \infty$, a direct straight line flight from r_0 to r_1.

For the values of $\cot(\gamma_0)$ from d to g , the conic goes from the elliptic to the hyperbolic region. <u>These two regions are divided by the parabola for which $\cot(\gamma_0) = -d$</u>.

Figure and Variable Definitions



t = time of flight from $\bar{r}_0$ to $\bar{r}_1$

α = $\frac{1}{a}$ where a = semimajor axis

p = semi-latus rectum

x = Herrick's variable, $\frac{\Delta E}{\sqrt{\alpha}}$ ellipse, $\frac{\Delta G}{\sqrt{-\alpha}}$ hyperbola

$S(Aug) = \frac{1}{3!} - \frac{Aug}{5!} + \frac{Aug^2}{7!} - \ldots.$ (Battin's Transcendental function)

$C(Aug) = \frac{1}{2!} - \frac{Aug}{4!} + \frac{Aug^2}{6!} - \ldots.$ (Battin's Transcendental function)

μ = gravitational constant

r_p = pericenter radius

For convenience the following nondimensional variables are defined:

$R = r_0/r_1$ $\qquad$ $P = p/r_0$

$X = x/\sqrt{r_0}$ $\qquad$ $T = \sqrt{\frac{\mu}{r_0^3}}\, t$

$A = r_0\alpha$ $\qquad$ $V = \sqrt{\frac{r_0}{\mu}}\, v_0$

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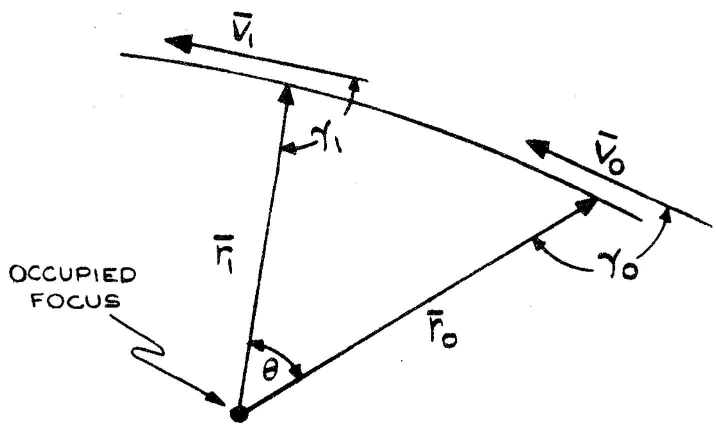

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Definition of Those Variables Used in the Appendix Not Previously Defined.

ΔE	=	eccentric anomaly difference (ellipse)*
ΔG	=	hyperbolic equivalent to ΔE*
e	=	eccentricity
f	=	true anomaly (measured from pericenter)
v_r	=	radial velocity
v_c	=	circumferential velocity
θ	=	true anomaly difference $(f_1 - f_0)$
h	=	angular momentum

* See references for definitions.