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Space Guidance Analysis Memo #44

TO: SGA Distribution

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SUBJECT: Optimization of Midcourse Correction Times

Section 4 of reference [1] discusses the possibility of varying the times of midcourse velocity corrections to minimize a function $\phi$, which involves the terminal deviation variances and a statistical measure of the total velocity change used for control.

It is suggested that $\partial\phi/\partial t_j$ be calculated by changing $t_j$ to $t_j + \Delta t_j$ and recalculating the X(t) and E(t) histories. In such an approach $\Delta t_j$ would have to be kept small to make the assumed linearization valid. But when $\Delta t$ is small, the computation of the derivative inevitably involves taking differences of nearly-equal numbers, a procedure to be avoided if reasonable accuracy is to be attained. The following procedure allows direct computation of the derivative.

A general performance criterion of this type is

$$\phi = \sum_{i=1}^{n} \operatorname{tr} f_i \left( X(t_i -),\ E(t_i -) \right) = \min.$$

where n - 1 corrections are made and $t_n$ is the final time.

The basic problem is that of finding the partial derivatives

$$\frac{\partial X(t_i -)}{\partial t_k} \quad \text{and} \quad \frac{\partial E(t_i -)}{\partial t_k} \qquad (k \leq i \leq n)$$

We first derive the relations describing the propagation of perturbations in the variance matrices, $\delta X(t)$ and $\delta E(t)$. From (3-1)* and (4-7), with $\Lambda = Q = 0$ and $K = EH^T R^{-1}$:

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*Hyphenated equation numbers refer to reference [1]

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$$\dot{X} = XF^T + FX \tag{1}$$

$$\dot{E} = EF^T + FE - EH^T R^{-1} HE \tag{2}$$

Equation (1) is linear and the corresponding perturbation equation, from (4-29), has the solution

$$\delta X_{i+1-} = \Phi_{i+1,\,i}\, \delta X_{i+}\, \Phi_{i+1,\,i}^T \qquad (i > k) \tag{3}$$

Equation (2) is of the Riccati type, and its perturbed solution (see Appendix A) satisfies the equation

$$\delta E_{i+1-} = \left[E_{i+1-}\, \Phi_{i+1,\,i}^{-T}\, E_{i+}^{-1}\right] \delta E_{i+} \left[E_{i+}^{-1}\, \Phi_{i+1,\,i}^{-1}\, E_{i+1-}\right] \qquad (i > k) \tag{4}$$

From Eq. (4-33),

$$E_{i+} = E_{i-} + G_i Q_i G_i^T \tag{4-33}$$

where Q is the control implementation error covariance matrix, we see that

$$\delta E_{i+} = \delta E_{i-} + G_i\, \delta Q_i G_i^T$$

where $\delta Q_i$ can be expressed in terms of $\delta X_{i-}$ and $\delta E_{i-}$, or may be zero. Combining this with Eq. (4) we can write $\delta E_{i+1-}$ in the form

$$\delta E_{i+1-} = \sum_j N_{xij}\, \delta X_{i-}\, N'_{xij} + \sum_j N_{eij}\, \delta E_{i-}\, N'_{eij} \tag{5}$$

The propagation of $\delta X$ across a correction is derived by perturbing Eq. (4-28)

$$X_{i+} = X_{i-} + G_i \left[B(X-E)B^T + Q\right]_{i-} G_i^T + \left[(X-E)B^T G^T + GB(X-E)\right]_{i-} \tag{4-28}$$

where B (derived by Battin) determines the correction $B\,\delta\hat{x}$.

$$\delta X_{i+} = \delta X_{i-} + G_i B_i \delta X_{i-} B_i^T G_i^T$$
$$+ \quad \delta X_{i-} B_i^T G_i^T + \quad G_i B_i \delta X_{i-}$$
$$- G_i B_i \delta E_{i-} B_i^T G_i^T - \quad \delta E_{i-} B_i^T G_i^T - \quad G_i B_i \delta E_{i-} + G_i \delta Q_i G_i^T \quad (6)$$

Combination of Eq. (3) with Eq. (6) gives an expression for $\delta X_{i+1-}$ of the form

$$\delta X_{i+1-} = \sum_j M_{xij} \; \delta X_{i-} \; M'_{xij} - \sum_j M_{eij} \; \delta E_{i-} \; M'_{eij} \tag{7}$$

The above recurrence formulas Eq. (5) and (7) require, as initial conditions, the values of $\delta E(t_{k+1}-)$ and $\delta X(t_{k+1}-)$, where $t_k$ is the correction time to be varied. By differentiating Eq. (4-28) with respect to time at $t_k$ we can express $\partial X(t_k+)/\partial t_k$ in terms of the time derivatives, at $(t_k-)$, of the components of the right hand side. $\dot{X}(t_k-)$ is obtained from Eq. (1), $\dot{G}$ from the original differential equations, $\dot{E}$ from Eq. (2), $\dot{B}$ from Battin's derivation of B, and $\dot{Q}$ from the (time-varying) statistical properties of the control implementation error v. If Q is also a function of X and E, its derivative must be calculated accordingly. We can thus write

$$\delta X(t_k+) = M_{k+} \, \delta t_k \tag{8}$$

Now Eq. (3) must be altered, for i = k, because $\Phi_{k+1,\,k}$ changes as we vary $t_k$. Hence

$$\delta X_{k+1-} = \Phi_{k+1,\,k} \, \delta X_{k+} \Phi^T_{k+1,\,k} + 2 \, \Phi_{k+1,\,k} X_{k+} \, \delta \Phi^T_{k+1,\,k} \tag{9}$$

Now the transition matrix may be written

$$\Phi_{k+1,\,k} = \Gamma(t_{k+1}) \, \Gamma^{-1}(t_k) \tag{10}$$

where

$$\dot{\Gamma}(t) = F\,\Gamma(t)$$

Hence

$$\frac{\partial}{\partial t_k} \Phi_{k+1,\,k} = \Gamma(t_{k+1}) \frac{d}{d\,t_k} \Gamma^{-1}(t_k) = - \Gamma_{k+1}\Gamma_k^{-1}\dot{\Gamma}_k\Gamma_k^{-1}$$

$$= - \Phi_{k+1,\,k}\, F\,(t_k) \tag{11}$$

Hence, combining Eqs. (8), (9) and (11) we may write

$$\delta X\,(t_{k+1}-) = M_{k+1}\,\delta t_k \tag{12}$$

In similar fashion, by differentiating (4-33) we can express $\delta E\,(t_k+)$ as

$$\delta E\,(t_k+) = N_{k+}\,\delta t_k \tag{13}$$

The relation between $\delta E\,(t_k+)$ and $\delta E\,(t_{k+1}-)$ is derived in Appendix B, and allows us to write

$$\delta E\,(t_{k+1}-) = N_{k+1}\,\delta t_k \tag{14}$$

With Eqs. (12) and (14) and the recurrence formulas Eqs.(5) and (7) we can now evaluate the matrices $M_i$ and $N_i$ $(i > k+1)$ which enable us to write

$$\delta X\,(t_i-) = M_i\,\delta t_k \qquad (i > k) \tag{15}$$

$$\delta E\,(t_i-) = N_i\,\delta t_k \qquad (i > k) \tag{16}$$

The matrices $M_k$ and $N_k$ are just the time derivatives of X and E at $t_k$, as determined from Eqs. (1) and (2).

# Appendices

## The Perturbed Riccati Equation

### A. Fixed Initial and Final Times.

The solution of equation (2) is given by Kalman [2] as

$$E = \left[\theta_{21} + \theta_{22} E_0\right] \left[\theta_{11} + \theta_{12} E_0\right]^{-1} \qquad \text{(A1)}$$

$E_0$ and E here will represent values at the beginning and end of an interval between corrections, $t_{i+}$ and $t_{i+1-}$. The matrix

$$\theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \qquad \text{(A2)}$$

is the transition matrix of

$$\dot{\underline{x}} = - F^T \underline{x} + H^T R^{-1} H\underline{w} \qquad \text{(A3)}$$

$$\dot{\underline{w}} = F\underline{w} \qquad \text{(A4)}$$

Obviously, $\theta_{21} = 0$ in this case and $\theta_{22} = \Phi_{i+1, i}$. $\theta_{11}$ is the transition matrix of the equation (A3), adjoint to (A4), and can be shown to be

$$\theta_{11} = \Phi_{i+1, i}^{-T} \qquad \text{(A5)}$$

From Eq. (A1) we can deduce that, with $\theta_{21} = 0$,

$$\theta_{12} = E^{-1} \theta_{22} - \theta_{11} E_0^{-1} \qquad \text{(A6)}$$

By perturbing Eq. (A1) we find

$$\delta E = \left[\theta_{22}\delta E_0\right]\left[\theta_{11} + \theta_{12}E_0\right]^{-1}$$

$$- \left[\theta_{21} + \theta_{22}E_0\right]\left[\theta_{11} + \theta_{12}E_0\right]^{-1} \theta_{12}\delta E_0\left[\theta_{11} + \theta_{12}E_0\right]^{-1}$$

$$= \left[\theta_{22} - E\theta_{12}\right] \delta E_0\left[\theta_{11} + \theta_{12}E_0\right]^{-1} \qquad \text{(A7)}$$

With $\theta_{21} = 0$, this can be written, using Eq. (A1),

$$\delta E = \left[\theta_{22} - E\theta_{12}\right] \delta E_0\left[\theta_{22}E_0\right]^{-1} E \qquad \text{(A8)}$$

which, together with Eq. (A6), gives

$$\delta E = \left[E\Phi^{-T}E_0^{-1}\right] \delta E_0\left[E_0^{-1}\Phi^{-1}E\right] \qquad \text{(A9)}$$

B. Variable Initial Time.

We now consider the effect of varying the initial time on the Eq. (A9). We consider only the special case $\theta_{21} = 0$. $E_0$ and E represent $E_{k+}$ and $E_{k+1-}$. Equation (A1) can be written

$$E = \Phi E_0\left[\Phi^{-T} + \theta_{12}E_0\right]^{-1} \qquad \text{(B1)}$$

and Eq. (A6) can be written

$$\theta_{12} = E^{-1}\Phi - \Phi^{-T}E_0^{-1} \qquad \text{(B2)}$$

Hence, from Eq. (B1),

$$\delta E = \left[\delta\Phi E_0 + \Phi\delta E_0\right]\left[\Phi^{-T} + \theta_{12}E_0\right]^{-1}$$

$$- E\left[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0\right]\left[\Phi^{-T} + \theta_{12}E_0\right]^{-1}$$

$$= \left[\delta\Phi E_0 + \Phi\delta E_0\right] E_0^{-1}\Phi^{-1}E$$

$$- E\left[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0\right] E_0^{-1}\Phi^{-1}E \qquad \text{(B3)}$$

where, using Eq. (11),

$$\delta \Phi = -\Phi F_k \delta t_k \tag{B4}$$

$$\delta \Phi^{-T} = -\Phi^{-T} \delta \Phi^T \Phi^{-T} = \Phi^{-T} F_k^T \delta t_k \tag{B5}$$

The quantity $\delta\theta_{12}$ is determined by noting that the solution of Eqs. (A3) and (A4) can be expressed as

$$\underline{w}(t) = \Phi(t, t_k)\underline{w}(t_k) \tag{B6}$$

$$\underline{x}(t) = \Phi^{-T}(t, t_k)\underline{x}(t_k) + \int_{t_k}^{t} \Phi^{-T}(t, \tau)H^T(\tau)R^{-1}(\tau)H(\tau)\underline{w}(\tau)\, d\tau \tag{B7}$$

Since the last term represents $\theta_{12}(t, t_k)\underline{w}(t_k)$, it follows that

$$\theta_{12}(t, t_k) = \int_{t_k}^{t} \Phi^{-T}(t, \tau)H^T(\tau)R^{-1}(\tau)H(\tau)\Phi(\tau, t_k)d\tau \tag{B8}$$

from which

$$\frac{\partial}{\partial t_k}\theta_{12}(t_{k+1}, t_k) = -\Phi^{-T}H_k^T R_k^{-1} H_k + \int_{t_k}^{t_{k+1}} \Phi^{-T}(t_{k+1}, \tau)H^T(\tau)R^{-1}(\tau)H(\tau)\frac{\partial}{\partial t_k}\Phi(\tau, t_k)d\tau$$

$$= -\Phi^{-T}H_k^T R_k^{-1} H_k - \theta_{12}(t_{k+1}, t_k)F_k \tag{B9}$$

where we have made use of Eqs. (B4) and (B8) in evaluating the last term.

Combining Eqs. (B2) - (B5), (B9) and (13), and simplifying,

$$\delta E = E\Phi^{-T}\left[H_k^T R_k^{-1} H_k + E_0^{-1} N_{k+} E_0^{-1} - F_k^T E_0^{-1} - E_0^{-1} F_k\right]\Phi^{-1} E\, \delta t_k \tag{B10}$$

## References

[1] Denham, W. F., and Speyer, J. L., "Optimal Measurement and Velocity Correction Programs for Midcourse Guidance," Raytheon report BR-2386, April 24, 1963.

[2] Kalman, R. E., and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," ASME Journal of Basic Engineering, March 1961.