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Space Guidance Analysis Memo # 38-64

TO: SGA Distribution
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SUBJECT: Optimal Guidance Laws

The material in this memo is taken primarily from lectures given by Professor Bryson. A general guidance law is presented and then it is applied to

(a) injection
(b) interception
(c) rendezvous
(d) soft landing

Let subscript p denote the pursuing body and T the target body.

The equations of motion are

$$\dot{\underline{v}}_T = \underline{f}_T$$

where $\underline{f}_T$ = external forces per unit mass

$$\dot{\underline{v}}_p = \underline{f}_p + \underline{a}$$

where $\underline{a}$ = control acceleration

$$\dot{\underline{r}}_T = \underline{v}_T$$

$$\dot{\underline{r}}_p = \underline{v}_p$$

We now have 12 equations, but we want relative values, consequently the 12 equations reduce to 6 by letting

$$\underline{v} = \underline{v}_p - \underline{v}_T$$

$$\underline{r} = \underline{r}_p - \underline{r}_T$$

$$\underline{f} = \underline{f}_p - \underline{f}_T$$

The 6 equations are

$$\dot{\underline{v}} = \underline{f} + \underline{a}$$
$$\dot{\underline{r}} = \underline{v} \qquad t_0 \leqq t \leqq t_1$$

where t_0 is initial time and t_1 is final time. The problem is to find the control law $\underline{a}$ which minimizes the cost function J, where J is assumed to be

$$J = \frac{1}{2}\left[C_v \, \underline{v} \cdot \underline{v} + C_r \, \underline{r} \cdot \underline{r}\right]_{t_1} + \int_{t_0}^{t_1} \frac{a^2}{2} \, dt$$

and which satisfies the differential equations

$$\dot{\underline{v}} = \underline{f} + \underline{a}$$

$$\dot{\underline{r}} = \underline{v}$$

This is a control problem to minimize J, the cost function, with initial state, time and final time fixed. The cost function J considers fuel cost and C_r, C_v are weighting factor constants for position and velocity. Form the Hamiltonian

$$H = \frac{a^2}{2} + \underline{\lambda} \cdot \underline{v} + \underline{p} \cdot (\underline{f} + \underline{a})$$

and consider the Hamiltonian system (differential equations associated with H)

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{v}} = -\underline{\lambda}$$

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{r}} = 0 \rightarrow \underline{\lambda} = \text{constant}$$

$\underline{p}$, $\underline{\lambda}$ are the adjoint free vectors.

For optimality (special case of maximum principle)

$$\frac{\partial H}{\partial \underline{a}} = 0 = \underline{a} + \underline{p} \rightarrow \underline{a} = -\underline{p}$$

$$\underline{p} = -(t - t_0)\underline{\lambda} + \underline{p}_0$$

solving for $\underline{a}$

$$\underline{a} = (t - t_0)\underline{\lambda} - \underline{p}_0 \qquad (1)$$

$$\dot{\underline{v}} = \underline{f} + (t - t_0)\underline{\lambda} - \underline{p}_0$$

$$\dot{\underline{r}} = \underline{v}$$

from transversality conditions

$$\underline{p}(t_1) = C_v \underline{v}(t_1)$$

$$\underline{\lambda}(t_1) = C_r \underline{r}(t_1)$$

let

$$\underline{v}(t_0) = \underline{v}_0$$

$$\underline{r}(t_0) = \underline{r}_0$$

solve for $\underline{v}$, $\underline{r}$ with $\underline{f}$ = constant

$$\underline{v} = (t - t_0)\underline{f} + \frac{(t - t_0)^2}{2} \underline{\lambda} - \underline{p}_0 (t - t_0) + \underline{v}_0 \qquad (2)$$

$$\underline{r} = \frac{(t - t_0)^2}{2} \underline{f} + \frac{(t - t_0)^3}{6} \underline{\lambda} - \underline{p}_0 \frac{(t - t_0)^2}{2} + \underline{v}_0 (t - t_0) + \underline{r}_0 \qquad (3)$$

$\underline{\lambda}$ is constant, therefore

$$\underline{\lambda} = C_r \underline{r} (t_1) \qquad (4)$$

and

$$\underline{p} (t_1) = C_v \underline{v} (t_1) = - (t_1 - t_0) \underline{\lambda} + \underline{p}_0 \qquad (5)$$

Eqs. 1 to 5 are the basic equations which will now be applied to some special cases.

(A) Injection

Final position is not important, therefore

$$C_r = 0$$

$$\therefore \ \underline{\lambda} = 0$$

Therefore from (1)

$$\underline{a} = - \underline{p}_0$$

from (2)

$$\underline{v} (t_1) = (t_1 - t_0)\underline{f} - \underline{p}_0 (t_1 - t_0) + \underline{v}_0$$

$$\underline{v}_0 = \underline{v}_{p_0} - \underline{v}_T$$

$$\underline{v}(t_1) = (t_1 - t_0)\underline{f} - \underline{p}_0 (t_1 - t_0) + \underline{v}_{p_0} - \underline{v}_T$$

$$\underline{v}(t_1) + \underline{p}_0 (t_1 - t_0) = (t_1 - t_0)\underline{f} + \underline{v}_{p_0} - \underline{v}_T$$

from (5)

$$p(t_1) = C_v \underline{v}(t_1) = -(t_1 - t_0)\underline{\lambda} + \underline{p}_0 = \underline{p}_0$$

since

$$\underline{\lambda} = 0$$

therefore $\underline{p}$ is constant.

$$\underline{p}_0 = \frac{\underline{f}(t_1 - t_0) + \underline{v}_{p_0} - \underline{v}_T}{(t_1 - t_0) + \frac{1}{C_v}}$$

$$\underline{f} = \underline{f}_p - \underline{f}_T = \underline{g} - 0 = \text{constant}$$

Usually want

$$\underline{v}(t_1) = 0$$

then $$C_v \rightarrow \infty$$

$$\boxed{\underline{a}(t_1) = \underline{p}_0 = -\underline{g} - \frac{\underline{v}_{p_0} - \underline{v}_T}{t_1 - t_0}}$$

sample data feedback law where t_0 is latest sample time.

for continuous measurements

$$\underline{a} = -\underline{g} - \frac{\underline{v}_p - \underline{v}_T}{t_1 - t}$$

continuous feedback control law

(**B**) Interception

For interception, final velocity is not important, therefore

$$C_v = 0$$

$$\underline{a} = (t - t_0)\,\underline{\lambda} - \underline{p}_0$$

Therefore, one must determine $\underline{\lambda}$, $\underline{p}_0$

Assume $\underline{f} = 0$

$$\underline{\lambda} = C_r \left[\frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \underline{p}_0 \frac{(t_1 - t_0)^2}{2} + \underline{v}_0 (t_1 - t_0) + \underline{r}_0 \right]$$

$$\underline{p}_0 = (t_1 - t_0)\,\underline{\lambda}$$

$$\underline{\lambda} = C_r \left[\frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \frac{(t_1 - t_0)^3}{2}\,\underline{\lambda} + \underline{v}_0 (t_1 - t_0) + \underline{r}_0 \right]$$

$$\underline{\lambda} - C_r \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} + C_r \frac{(t_1 - t_0)^3}{2}\,\underline{\lambda} = C_r \left[\underline{v}_0 (t_1 - t_0) + \underline{r}_0 \right]$$

$$\frac{\underline{\lambda}}{C_r} - \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} + \frac{(t_1 - t_0)^3}{2}\,\underline{\lambda} = \underline{v}_0 (t_1 - t_0) + \underline{r}_0$$

Solving for $\underline{\lambda}$:

$$\underline{\lambda} = \frac{\underline{v}_0 (t_1 - t_0) + \underline{r}_0}{\dfrac{(t_1 - t_0)^3}{3} + \dfrac{1}{C_r}}$$

Solve for $\underline{p}_0$ in $\underline{p} = (t_1 - t_0)\,\underline{\lambda}$ to yield

$$\underline{p}_0 = \frac{(t_1 - t_0)\left[\underline{v}_0 (t_1 - t_0) + \underline{r}_0\right]}{\frac{(t_1 - t_0)^3}{3} + \frac{1}{C_r}}$$

$$\underline{a} = \frac{(t - t_0)\left[\underline{v}_0 (t_1 - t_0) + \underline{r}_0\right]}{\frac{(t_1 - t_0)^3}{3} + \frac{1}{C_r}} - \frac{(t_1 - t_0)\left[\underline{v}_0 (t_1 - t_0) + \underline{r}_0\right]}{\frac{(t_1 - t_0)^3}{3} + \frac{1}{C_r}}$$

Combining terms to yield

$$\underline{a} = - \frac{(t_1 - t)\left[\underline{v}_0 (t_1 - t_0) + \underline{r}_0\right]}{\frac{(t_1 - t_0)^3}{3} + \frac{1}{C_r}}$$ sample data case

$$\underline{a} = - \frac{(t_1 - t)\left[\underline{v} (t_1 - t) + \underline{r}\right]}{\frac{(t_1 - t)^3}{3} + \frac{1}{C_r}}$$ continuous feedback control law

NOTE:

As $C_r \rightarrow \infty$, $\underline{r}(t_1) \rightarrow 0$ at the expense of large $\int_{t_0}^{t} a^2 \, dt$

Or $\underline{a} \rightarrow \frac{-3\left[\underline{v}(t_1 - t) + \underline{r}\right]}{(t_1 - t)^2}$ as $C_r \rightarrow \infty$

(C) Rendezvous

To place pursuing body in immediate vicinity of target with small component of relative velocity.

$$\underline{a} = (t - t_0)\,\underline{\lambda} - \underline{p}_0$$

Assume $\underline{f} = 0$

This is reasonable if the distance from the target to pursuing body is small, since in this case

$$\underline{f}_p \approx \underline{f}_T$$

$$v(t_1) = 0 = \frac{(t_1 - t_0)^2}{2}\,\underline{\lambda} - \underline{p}_0\,(t_1 - t_0) + \underline{v}_0$$

$$r(t_1) = 0 = \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \underline{p}_0\,\frac{(t_1 - t_0)^2}{2} + \underline{v}_0\,(t_1 - t_0) + \underline{r}_0$$

$$\underline{p}_0\,(t_1 - t_0) = \underline{v}_0 + \frac{(t_1 - t_0)^2}{2}\,\underline{\lambda}$$

$$\underline{p}_0 = \frac{\underline{v}_0}{t_1 - t_0} + \frac{t_1 - t_0}{2}\,\underline{\lambda}$$

$$0 = \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \left[\frac{\underline{v}_0}{t_1 - t_0} + \frac{t_1 - t_0}{2}\,\underline{\lambda}\right]\frac{(t_1 - t_0)^2}{2} + \underline{v}_0\,(t_1 - t_0) + \underline{r}_0$$

$$0 = \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \underline{v}_0\,\frac{(t_1 - t_0)}{2} - \frac{(t_1 - t_0)^3}{4}\,\underline{\lambda} + \underline{v}_0\,(t_1 - t_0) + \underline{r}_0$$

$$-\frac{\underline{v}_0 (t_1 - t_0)}{2} - \underline{r}_0 = \frac{\underline{\lambda}}{12}\left[(t_1 - t_0)^3 \right]$$

$$\underline{\lambda} = \frac{-6\,\underline{v}_0}{(t_1 - t_0)^2} - \frac{12\,\underline{r}_0}{(t_1 - t_0)^3}$$

$$\underline{p}_0 = \frac{\underline{v}_0}{t_1 - t_0} + \frac{t_1 - t_0}{2}\left[-\frac{6\,\underline{v}_0}{(t_1 - t_0)^2} - \frac{12\,\underline{r}_0}{(t_1 - t_0)^3} \right]$$

$$\underline{p}_0 = \frac{\underline{v}_0}{t_1 - t_0} - \frac{3\,\underline{v}_0}{t_1 - t_0} - \frac{6\,\underline{r}_0}{(t_1 - t_0)^2}$$

$$\underline{p}_0 = -\frac{2\,\underline{v}_0}{t_1 - t_0} - \frac{6\,\underline{r}_0}{(t_1 - t_0)^2}$$

$$\underline{a} = (t - t_0)\left[\frac{-6\underline{v}_0}{(t_1 - t_0)^2} - \frac{12\,\underline{r}_0}{(t_1 - t_0)^3} \right] + \frac{2\,\underline{v}_0}{t_1 - t_0} + \frac{6\,\underline{r}_0}{(t_1 - t_0)^2}$$

$$\underline{a} = \frac{-6\,\underline{v}_0}{t_1 - t_0} - \frac{12\,\underline{r}_0}{(t_1 - t_0)^2} + \frac{2\,\underline{v}_0}{t_1 - t_0} + \frac{6\,\underline{r}_0}{(t_1 - t_0)^2}$$

$$\underline{a}(t) = \frac{-4\,\underline{v}_0}{t_1 - t_0} - \frac{6\,\underline{r}_0}{(t_1 - t_0)^2}$$

$$\boxed{\underline{a}(t) = \frac{-4\,v}{t_1 - t} - \frac{6\,r}{(t_1 - t)^2}}$$

(D) Soft Landing

Assume

$$\underline{f} = \underline{g} = \text{constant}$$

$$\underline{a} = (t - t_0)\,\underline{\lambda} - \underline{p}_0$$

$$0 = \frac{(t_1 - t_0)^2}{2}\,\underline{\lambda} - \underline{p}_0\,(t_1 - t_0) + \underline{v}_0 + (t_1 - t_0)\,\underline{g}$$

$$0 = \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \underline{p}_0\,\frac{(t_1 - t_0)^2}{2} + \underline{v}_0\,(t_1 - t_0) + \underline{r}_0 + \frac{(t_1 - t_0)^2}{2}\,\underline{g}$$

$$\underline{p}_0 = \frac{\underline{v}_0}{t_1 - t_0} + \frac{t_1 - t_0}{2}\,\underline{\lambda} + \underline{g}$$

$$0 = \frac{(t_1 - t_0)^3}{6}\,\underline{\lambda} - \frac{1}{2}\left[\frac{\underline{v}_0}{t_1 - t_0} + \frac{t_1 - t_0}{2}\,\underline{\lambda} + \underline{g}\right](t_1 - t_0)^2 + \underline{v}_0\,(t_1 - t_0)$$

Solve for $\underline{p}_0$, $\underline{\lambda}$ and substitute in equation for $\underline{a}$ to yield

$$\underline{a} = \frac{-4\,\underline{v}}{t_1 - t} - \frac{6\,\underline{r}}{(t_1 - t)^2} - \underline{g}$$