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Space Guidance Analysis Memo #29

To: SGA Distribution

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Date: November 20, 1962

Subject: Error Ellipsoids

In this memo the definition of error ellipsoids for Gaussian distributions in n-space is given. A formula is developed for evaluating the "size parameter" K for the ellipsoid corresponding to a given confidence level in terms of exponential functions and error functions. A numerical technique for applying this formula in the six dimensional case is discussed. Expressions for the principal axes of the error ellipsoid in terms of K and the eigenvalues and eigenvectors of the covariance matrix are obtained and a formula for the volume of the error ellipsoid corresponding to a given confidence level is derived.

Let $\underline{\xi}$ be a random n-dimensional column vector with a Gaussian distribution and let C be its covariance matrix, that is

$$C = E\left[\underline{\xi}\,\underline{\xi}^t\right]. \tag{1}$$

Furthermore, assume that $\underline{\xi}$ has zero mean and that C is non-degenerate in the sense that it is positive definite and not merely positive semi-definite. Then the probability density for $\underline{\xi}$ in n-space is

$$P(x) = \alpha\, e^{-(1/2)\underline{x}^t C^{-1} \underline{x}}. \tag{2}$$

Surfaces of constant probability density correspond to the loci of points $\underline{x}$ for which

$$\underline{x}^t C^{-1} \underline{x} = K \tag{3}$$

where K is a constant. Let $B = C^{-1}$. Since C is positive definite, its inverse B is positive definite and equation (3) defines an n-dimensional ellipsoid, E_K.

As the constant K in equation (3) increases, the ellipsoid E_K expands and the probability that the tip of the random vector $\underline{\xi}$ lies inside E_K increases toward one. The error ellipsoid corresponding to the probability p is the ellipsoid E_K for which the probability that $\underline{\xi}$ lies inside E_K is p.

FORMULA FOR K

The probability that x lies inside the ellipsoid E_K is

$$P_K = \int_{E_K} \alpha \, e \, \exp\left[-\frac{1}{2}\underline{x}^t B \underline{x}\right] d\underline{x} \tag{4}$$

where $d\underline{x}$ represents an n-dimensional volume element. Since C is positive definite, there is a real symmetric matrix A (the "square root" of C) such that

$$C = A^2 \tag{5}$$

Let

$$\underline{y} = A^{-1} \underline{x} \quad .$$

Then

$$\underline{x}^t B \underline{x} = \underline{y}^t A B A \underline{y}$$

but

$$B = C^{-1} = A^{-1} A^{-1}$$

or

$$\underline{x}^t B \underline{x} = \underline{y}^t \underline{y} \quad . \tag{6}$$

Thus the ellipse E_K in x-space corresponds to the sphere

$$\underline{y}^t \underline{y} = K$$

in y-space. Transformed to y-space, the integral (4) becomes

$$P_K = \int_{\underline{y}^t \underline{y} \leq K} \alpha\, e \exp\left[-\frac{1}{2}\, \underline{y}^t \underline{y} \right] \beta\, d\underline{y} \tag{7}$$

where β is the Jacobian $\dfrac{\partial(\underline{x})}{\partial(\underline{y})} = \left|\det(A)\right|$. The area of a spherical shell of radius r in n-space is

$$\gamma\ r^{n-1} \tag{8}$$

where γ is a constant depending on the dimension n. If $\underline{y}$ lies on a spherical shell of radius r,

$$\underline{y}^t\, \underline{y} = r^2. \tag{9}$$

Using equations (8) and (9) to transform the integral (7) to spherical coordinates we have

$$P_K = \int_0^{\sqrt{K}} \alpha\beta\gamma\ e^{-(1/2)r^2}\, r^{n-1}\, dr \quad . \tag{10}$$

Since $P_\infty = 1$, equation (10) can be rewritten

$$P_K = \frac{\displaystyle\int_0^{\sqrt{K}} \alpha\beta\gamma\ e^{-(1/2)r^2}\, r^{n-1}\, dr}{\displaystyle\int_0^{\infty} \alpha\beta\gamma\ e^{-(1/2)r^2}\, r^{n-1}\, dr} \tag{11}$$

or cancelling the $\alpha\beta\gamma$ factors in the numerator and denominator of (11)

$$P_K = \frac{\displaystyle\int_0^{\sqrt{K}} e^{-(1/2)r^2}\, r^{n-1}\, dr}{\displaystyle\int_0^{\infty} e^{-(1/2)r^2}\, r^{n-1}\, dr} \quad . \tag{12}$$

Evaluation of the integral

$$F_n(K) = \int_0^{\sqrt{K}} e^{-(1/2)r^2} \, r^{n-1} \, dr$$

can be reduced to the evaluation of

$$F_2(K) = \int_0^{\sqrt{K}} e^{-(1/2)r^2} \, r \, dr$$

$$= 1 - e^{-K/2}$$

if n is even or

$$F_1(K) = \int_0^{\sqrt{K}} e^{-(1/2)r^2} \, dr$$

$$= \sqrt{\frac{\pi}{2}} \; \mathrm{erf}\left(\sqrt{\frac{K}{2}}\right)$$

if n is odd by using the recursion formula

$$F_n(K) = (n-2)\, F_{(n-2)}(K) - K^{\frac{n-2}{n}} e^{-K/2} . \tag{13}$$

Equation (13) is obtained by integration by parts as follows:

$$F_n(K) = \int_0^{\sqrt{K}} e^{-(1/2)r^2} \, r^{n-1} \, dr$$

$$= \int_0^{\sqrt{K}} - r^{n-2} \, d\left(e^{-(1/2)r^2}\right)$$

$$= - r^{n-2} e^{-(1/2) r^2} \Big|_0^{\sqrt{K}}$$

$$+ (n-2) \int_0^{\sqrt{K}} r^{n-3} e^{-(1/2) r^2} \, dr$$

$$= (n-2) F_{n-2}(K) - K^{\frac{n-2}{2}} e^{-(1/2) K} .$$

In particular

$$F_6 (K) = 8 - (8 + 4K + K^2) \, e^{-(1/2) K} \tag{14}$$

Thus when n = 6

$$P_K = 1 - (1 + \tfrac{1}{2} K + \tfrac{1}{8} K^2) \, e^{-(1/2) K} \tag{15}$$

since $F_6(\infty) = 8$. From equation (12) it follows that P_K is a strictly increasing function of K and that, as K increases from zero to infinity, P_K increases from zero to one. Thus for any value of P between zero and one there is exactly one value of K such that $P_K = P$.

For numerical evaluation of $F_1(K)$ when K is larger than nine or ten it is probably more convenient to use an asymptotic series rather than an error function table. The error committed in using the following expansion is less than the first term dropped.

$$F_1(K) \sim \sqrt{\frac{\pi}{2}} - \frac{e^{-(1/2) K}}{\sqrt{K}} \left[1 - \frac{1}{K} + \frac{1 \cdot 3}{K^2} - \frac{1 \cdot 3 \cdot 5}{K^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{K^4} - \dots \right]$$

NUMERICAL TECHNIQUE

Assuming that $n = 6$ and writting $P = 1 - \epsilon$, we have

$$\epsilon = (1 + \frac{1}{2} K + \frac{1}{8} K^2) e^{-(1/2)K}. \tag{16}$$

To find the value of K corresponding to the confidence level ϵ it is necessary to solve equation (16) for K. Since the right hand side of equation (16) is a rapidly varying function of K, Newton's method will be more efficient if applied to the equation

$$0 = f(K) = K - 2 \log \left\{ \frac{1 + \frac{1}{2} K + \frac{1}{8} K^2}{\epsilon} \right\} .$$

Upon simplification, the recursion formula

$$K_{\ell + 1} = K_\ell - \frac{f(K_\ell)}{f'(K_\ell)}$$

becomes

$$K_{\ell + 1} = - \left[4 + \frac{8}{K_\ell} \right] + 2 \left[1 + \frac{4}{K_\ell} + \frac{8}{K_\ell^2} \right] A_\ell$$

where

$$A_\ell = \left[\log \left(1 + \frac{1}{2} K_\ell + \frac{1}{8} K_\ell^2 \right) + \log \frac{1}{\epsilon} \right] . \tag{17}$$

From equation (16) it follows that

$$\frac{-\log \epsilon}{K} = \frac{1}{2} - \frac{\log\left(1 + \frac{1}{2} K + \frac{1}{8} K^2\right)}{K}$$

and therefore

$$\frac{\log \frac{1}{\epsilon}}{K} \to \frac{1}{2}$$

as $K \to \infty$ (or equivalently, as $\epsilon \to 0$). Thus, a reasonable initial guess for the iteration (17) is

$$K_0 = 2 \log \frac{1}{\epsilon} \quad . \tag{18}$$

If $\epsilon = 0.01$ the resulting K_ℓ's are

$$K_0 = 9.210$$
$$K_1 = 17.727$$
$$K_2 = 16.817$$
$$K_3 = 16.812$$

and

$$\left(1 + \frac{1}{2} K_3 + \frac{1}{8} K_3^2\right) e^{-K_3/2} = 9.999 \times 10^{-3} \, .$$

PRINCIPAL AXES

Consider the problem of finding stationary values of $f(\underline{x}) = \sqrt{\underline{x}^t \underline{x}}$ (the distance from the origin) when $\underline{x}$ is constrained to lie on E_K. The vectors $\pm \underline{x}_\ell$ ($\ell = 1, \ldots, n$) for which $f(\underline{x})$ is stationary are called the principal axes of the ellipsoid E_K. By using Lagrange multipliers it follows easily that the $\underline{x}_\ell$'s are the eigenvectors of B. Since $C^{-1} = B$ and C have the same eigenvectors, the principle axes of E_K are the eigenvectors of C. Thus we have

$$C \underline{x}_\ell = \lambda_\ell \underline{x}_\ell$$

$$\underline{x}_\ell = \lambda_\ell C^{-1} \underline{x}_\ell$$

and

$$\underline{x}_\ell^t \underline{x}_\ell = \lambda_\ell \underline{x}_\ell^t C^{-1} \underline{x}_\ell = \lambda_\ell K.$$

Therefore the principal axes of E_K are the eigenvectors of C normalized so that the length of $\underline{x}_\ell$ is $\sqrt{K\lambda_\ell}$ where λ_ℓ is the eigenvalue corresponding to $\underline{x}_\ell$.

If $C_1, \ldots, C_n$ are scalars such that

$$\sum_{\ell = 1}^{n} C_\ell^2 = 1$$

then

$$\underline{x} = \sum_{\ell = 1}^{n} C_\ell \underline{x}_\ell \tag{19}$$

is a point on E_K. Something like polar coordinates can be introduced on E_K by using the formula

$$\underline{x} = \cos\theta_1 \underline{x}_1 + \sin\theta_1 \cos\theta_2 \underline{x}_2 + \sin\theta_1 \sin\theta_2 \cos\theta_3 \underline{x}_3$$

$$+ \cdots + \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \cos\theta_n \underline{x}_n$$

obtained from equation (19)by using the identity

$$\sin^2\theta + \cos^2\theta = 1$$

VOLUME OF ELLIPSOID

A single number which conveys some information about the error distribution is the volume of the error ellipsoid E_K corresponding to the "size parameter" K, or confidence level P. The volume of an n-dimensional ellipsoid is given by the formula

$$V = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} M_1 M_2 , , , M_n$$

where M_1 through M_n are the lenghts of the principal axes and Γ represents the gamma function. Since

$$M_\ell = \sqrt{K\lambda_\ell}$$

we have

$$V = \frac{(K\pi)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} (\lambda_1 \lambda_2 \cdots \lambda_n)^{1/2}$$

where $\lambda_1, , , , \lambda_n$ are the eigenvalues of C.

However

$$\lambda_1 \lambda_2 \cdots \lambda = \det C$$

so

$$V = \frac{(K\pi)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} (\det C)^{1/2}$$

Note that if C is singular so that the error vectors satisfy a linear relation, then V is zero.