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Space Guidance Analysis Memo #28-65

TO: SGA Distribution  
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SUBJECT: Conic Time of Flight Calculations to Radius of Specified Length

Introductory Comments

We wish to calculate the time to traverse an elliptic arc that starts at an arbitrary radius  $\bar{r}_0$  and goes in the direction of velocity  $\bar{v}_0$  to a radius of specified length  $r_h$ . Based upon requirements of Flight 202, some ground rules for an AGC program capable of computing time of flight are:

- a. the program should be general, and not limited by the size of the transfer angle.
- b. the entire AGC calculation should require less than 100 ms.
- c. accuracies on the order of one second are acceptable.
- d. the resulting time of flight should have the same binary scale factor as the T3 register; that is, one low order bit equals 1 cs.

Point a is easily satisfied by solution of Kepler's equation for the time of flight  $t_f$  except that Point b is of equal importance and might preclude so general a solution for  $t_f$ . However, the speed requirement suggests that single precision arithmetic might be used where practical, a suggestion that makes calculation of an arc tangent an attractive possibility. In single precision, the error in the angle  $E/\pi$  as determined by the arctan is at best one bit,  $2^{-14}$ , and causes an error in  $t_f$  that is  $2^{-14}$  times the dimensional factor,  $\pi a \sqrt{a/\mu}$ , where  $\mu$  establishes the time units. Thus the computational accuracy to which  $t_f$  is known is the same as the accuracy to which the arctan is determined. But the value of the least significant bit in the  $t_f$  register depends on the maximum value that the register is called upon to contain, and is proportional to  $a \sqrt{a/\mu}$ . Computation should be carried out at a binary scale level consistent with both the maximum  $t_f$  and any relevant numerical approximations and should be independent of the scale level wanted by the user program.

## Equations for Elliptic Orbits

The equations to follow summarize the results of the Appendix. The time of flight for the elliptic orbit in the direction of velocity  $\bar{v}_0$  from radius  $\bar{r}_0$  to a radius of specified length  $r_h$  is provided by Kepler's equation. (See Ref. 1)

$$t_f = a\sqrt{\frac{a}{\mu}} [E - E_0 - e(\sin E - \sin E_0)] \quad (1)$$

where either of the following forms is valid

$$e(\sin E - \sin E_0) = \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a}} - \text{sgn}(\dot{r}_0) \sqrt{\frac{r_0}{a} \left(2 - \frac{r_0}{a}\right) - \frac{p}{a}} \quad (2.1)$$

$$= \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a}} - \frac{\bar{v}_0 \cdot \bar{r}_0}{\sqrt{a\mu}} \quad (2.2)$$

and where the quantity  $(E - E_0)$  is obtained from

$$\tan\left(\frac{E - E_0}{2}\right) = \frac{e(\sin E - \sin E_0)}{2 - \frac{r_0}{a} - \frac{r_h}{a}} \quad (3)$$

The two conic parameters that appear are supplied by

$$\bar{H} = \bar{r}_0 \times \bar{v}_0$$

$$p = \frac{\bar{H} \cdot \bar{H}}{\mu} \quad (4)$$

$$\frac{1}{a} = \frac{2}{r_0} - \frac{\bar{v}_0 \cdot \bar{v}_0}{\mu}$$

and are used to give  $p/a$ ,  $r_0/a$ ,  $r_h/a$ . For negative  $t_f$ , see Appendix, Summary I and II.

Equations (4) assume that  $r_0 = |\bar{r}_0|$  is available in AGC memory. Equation (2.1) is preferred for reentry on Flight 202 because in that application both  $\text{sgn}(\dot{r}_0)$  and  $\text{sgn}(\dot{r}_h)$  are known and are negative and permit faster computation, particularly if single precision square roots are used. In applications where the radial velocity of  $\bar{r}_0$  can have either sign, (2.2) is the better choice.

## Block I AGC Time Estimate

If we assume that none of the quantities in (4) are already available in

AGC memory, almost all of the time used in calculating  $t_f$ , 77 ms out of 93 ms, is invested in the AGC Interpreter language program for evaluating (4) and in establishing  $r_o/a$ ,  $r_h/a$ , and  $p/a$  in the push down list. Evaluation of (1) and (2) is done in basic AGC language and requires 7 ms. A final 9 ms (Interpreter) is expended in a  $t_f$  test to recognize if  $\bar{r}_o$  has passed  $r_h$ . It may be possible to trim a few ms here and there, but a major gain is unrealizable unless either  $p$  or  $1/a$  is already available in memory, or unless the products involving  $1/a$  are done in basic AGC language.

### Arc Tangent

The arctangent approximant given by the Hastings polynomial (see Ref. 2) of four terms is compatible with single precision arithmetic, and has a maximum error between the approximant and the function of 0.00008 radians. The polynomial is scaled to  $E/\pi$ , so the error is about 0.00003 or less than one bit of a single precision word. For angles in the range  $\pm 45^\circ$ , the approximant is a polynomial in  $x$ :  $y = f(x)$ , where  $x = N/D$  and is in the range  $(-1, 1)$ . To cover the full first quadrant, the approximant becomes  $y = \pi/4 - f(z)$ , where  $z = (x - 1)/(x + 1) = (N - D)/(N + D)$  and  $x$  is in the range  $(0, \infty)$ . The latter relation avoids the problem associated with zero denominators that would occur if  $x = N/D$  were used. However, the indeterminate case, corresponding to  $90^\circ$ , can arise and must be checked for.

Throughout most of the first quadrant, the arctan program uses the polynomial in  $z$ . This polynomial has the disadvantage that it produces a non-zero error in  $y$  at the extremes,  $0^\circ$  and  $90^\circ$ . The polynomial in  $x$  has no error at  $0^\circ$  and is used for small angles. The transition from  $x$  to  $z$  is made at about  $15^\circ$ , corresponding to one of the several values of  $y$  for which both polynomials exhibit the same error.

The calculation of the arctan as detailed below is based on the four term polynomial in Ref. 2. The program accepts  $N$  and  $D$ , where  $\tan E = N/D$ , and gives  $E$  in the range  $(0, \pi)$ .

$$E = \pi/4$$

$$N1 = \text{abs}(N), D1 = \text{abs}(D)$$

$$\text{If } (N1 + D1)/e^{-10^{-6}} \text{ neg, } E = 2E, \text{ return. (indet. case)}$$

$$z = (N1 - D1)/(N1 + D1)$$

$$\text{If } z + .573 \text{ neg, } z = N1/D1, E = 0 \quad (\text{transition case})$$

$$E = E + z (C_1 + z^2 (C_3 + z^2 (C_5 + z^2 C_7)))$$

If  $\text{sgn}(N D)$  neg,  $E = \pi - E$

return

and  $C_1 = .999215$ ,  $C_3 = -.3211819$ ,  $C_5 = .1462766$ ,  $C_7 = -.0389929$

### Numerical Results

Equations (1) through (4) were tested by a MAC program on the MH1800. In the test the terminal point  $r_h$  corresponds to 400,000 feet altitude and true anomaly  $f = 330^\circ$ . Time of flight  $t_f$  is calculated for  $\bar{r}_0$  and  $\bar{v}_0$  corresponding to selected values of  $f_0$  ranging from  $90^\circ$  to  $327^\circ$  for each of the following values of  $e$ : 0.3, 0.2, 0.1, 0.05, 0.01. These conditions yield values of  $t_f$  that range from nearly 7000 seconds to 40 seconds. The error between  $t_f$  and a carefully evaluated check solution in all cases is attributable to the arctan approximant and is 0.2 seconds or less. The error in  $t_f$  due to error  $\delta E$  in arctan is

$$t_f = 2 a \sqrt{\frac{a}{\mu}} \delta E$$

and  $\delta E$  is 0.00008 radians.

### Equations for Hyperbolic Orbits

The time of flight equations for hyperbolic orbits are similar to those for elliptic orbits and are summarized below. The derivation is not given, but it parallels the one given in the Appendix for elliptic orbits.

The time of flight  $t_f$  required to traverse a hyperbolic arc from an arbitrary radius  $\bar{r}_0$  to a radius of specified length  $r_h$  is provided by Kepler's equation (see Ref. 1).

$$t_f = a \sqrt{\frac{a}{\mu}} [ - (H - H_0) + e (\sinh H - \sinh H_0) ] \quad (5)$$

where either of the following can be used

$$e (\sinh H - \sinh H_0) = \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left( 2 + \frac{r_h}{a} \right) - \frac{p}{a}} - \text{sgn}(\dot{r}_0) \sqrt{\frac{r_0}{a} \left( 2 + \frac{r_0}{a} \right) - \frac{p}{a}} \quad (6.1)$$

$$= \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left( 2 + \frac{r_h}{a} \right) - \frac{p}{a}} - \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{a \mu}} \quad (6.2)$$

and the quantity  $(H - H_0)$  comes from

$$\tanh\left(\frac{H - H_0}{2}\right) = \frac{e(\sinh H - \sinh H_0)}{2 + \frac{r_0}{a} + \frac{r_h}{a}} \quad (7)$$

In these equations,  $1/a$  is a positive number.

The inverse hyperbolic tangent is calculated from

$$y = \tanh^{-1}(x) = \frac{1}{2} \ln(10) \log_{10}\left(\frac{1+x}{1-x}\right)$$

where the log term can be approximated by a Hastings polynomial (see Ref. 2)  $f(z) = \log(z)$  over the range  $1 \leq z \leq 10$  or equivalently  $0 \leq x < 0.818$ . For reference, the three term approximation is suitable in single precision and is

$$\log(z) = \frac{1}{2} + C_1 Q + C_3 Q^3 + C_5 Q^5$$

where

$$Q = \frac{z - \sqrt{10}}{z + \sqrt{10}} = \frac{C_0 + x}{1 + C_0 x}$$

and  $C_0 = -0.51949385$ ,  $C_1 = 0.8690286$ ,  $C_2 = 0.2773839$ ,  $C_3 = 0.2543275$ .

Using the signed  $1/a$  obtained in (4), the equations for hyperbolic and elliptic orbits can be combined into a single set, except for taking the inverse function; the proper inverse is selected by the sign of  $1/a$ .

## REFERENCES

1. Astronautical Guidance, R. H. Battin, general reference, Chapter 2.
2. Approximations for Digital Computers, C. Hastings, Jr.

## APPENDIX

### Elliptic Time of Flight to a Radius of Specified Length

The time required to traverse the elliptic arc from an initial radius  $\bar{r}_0$  to a terminal radius of specified magnitude  $r_h$  and in the direction of velocity  $\bar{v}_0$  is given by Kepler's equation \*(equations so marked are found in Chapter 2 of Astronautical Guidance, by R. H. Battin)

$$t = a \sqrt{\frac{a}{\mu}} [ E - E_0 - e (\sin E - \sin E_0) ] \quad (A-1)$$

where

$E_0$  is eccentric anomaly corresponding to  $\bar{r}_0$

$E$  is eccentric anomaly corresponding to  $r_h$

( $r_h$  is radius corresponding to desired altitude)

$e$  is eccentricity

$a$  is semi major axis

$\mu$  is gravitational constant

Method I - One approach is to calculate independently both  $E_0$  and  $E$ . The former is obtained from initial position and velocity using the identity\*

$$\tan \frac{E_0}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f_0}{2} = \sqrt{\frac{1-e}{1+e}} \left( \frac{\sin f_0}{1 + \cos f_0} \right) = \frac{\bar{v}_0 \cdot \bar{r}_0 / \sqrt{\mu a}}{(1+e) - \frac{r_0}{a}} \quad (A-2)$$

since\*

$$e r_0 \sin f_0 = \bar{v}_0 \cdot \bar{r}_0 \sqrt{\frac{p}{\mu}} \quad (A-2.1)$$

$$e r_0 \cos f_0 = p - r_0$$

and also

$$e r_0 (1 + \cos f_0) = a (1 - e) \left[ 1 + e - \frac{r_0}{a} \right]$$

For convenience in the following derivations, let

$$Q\left(\frac{r_h}{a}\right) \equiv \operatorname{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left( 2 - \frac{r_h}{a} \right) - \frac{p}{a}} \quad (A-3)$$

For  $E$  we obtain

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \left( \frac{\sin f}{1+\cos f} \right) = \frac{Q\left(\frac{r_h}{a}\right)}{1+e-\frac{r_h}{a}} \quad (\text{A-4})$$

since, squaring the cos f relation of (A-2.1) and using (A-3), there follows

$$e r_h \sin f = \sqrt{a p} Q\left(\frac{r_h}{a}\right) \quad (\text{A-5})$$

In calculating the E's, the arctan uses the convention that if  $\tan^{-1}(\ )$  has a negative value, replace  $\tan^{-1}(\ )$  by  $\pi + \tan^{-1}(\ )$ .

It is possible at this point to solve (A-1), since both  $E_0$  and E are known, by computing the two sines. Such action is unnecessary because both are expressible in terms of quantities already derived:

$$e (\sin E - \sin E_0) = \left[ Q\left(\frac{r_h}{a}\right) - \frac{\bar{v}_0 \cdot \bar{r}_0}{\sqrt{a \mu}} \right] \quad (\text{A-6})$$

This relationship arises in the following manner. The quantity  $E_0$  is expressed in terms of the identity\* below and (A-2.1) or (A-5)

$$\sin E_0 = \frac{\sqrt{1-e^2} \sin f_0}{1+e \cos f_0} = \frac{r_0 \sin f_0}{\sqrt{a p}} = \frac{\bar{v}_0 \cdot \bar{r}_0}{e \sqrt{a \mu}} \quad (\text{A-6.1})$$

$$\sin E = \frac{r_h \sin f}{\sqrt{a p}} = \left(\frac{1}{e}\right) Q\left(\frac{r_h}{a}\right) \quad (\text{A-6.2})$$

### Summary: Method I

The time required to traverse the elliptic arc from an initial radius  $\bar{r}_0$  to a terminal radius of specified length  $r_h$  and in the direction of velocity  $\bar{v}_0$  is given by Kepler's equation in the form

$$t = a \sqrt{\frac{a}{\mu}} \left[ E - E_0 - \operatorname{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a} + \frac{\bar{v}_0 \cdot \bar{r}_0}{\sqrt{a \mu}}} \right] \quad (\text{A-7.1})$$

where E and  $E_0$  come from

$$\tan \frac{E_0}{2} = \frac{\bar{v}_0 \cdot \bar{r}_0}{\sqrt{a \mu}} \left/ \left(1 + e - \frac{r_0}{a}\right) \right. \quad (\text{A-7.2})$$



$$\tan \frac{E}{2} = \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a}} \bigg/ \left(1 + e - \frac{r_h}{a}\right) \quad (\text{A-7.3})$$

These require three parameters  $a$ ,  $p$ ,  $e$ .  $\text{Sgn}(\dot{r}_h)$  designates which of the two vectors having length  $r_h$  is to be used. During reentry,  $\text{sgn}(\dot{r}_h)$  is negative. Both of the tangent equations are indeterminate when the true anomaly is  $180^\circ$  but correspond to an eccentric anomaly of  $180^\circ$ . This situation arises only for (A-7.2) and is recognized by the arctan. Equation (A-7.3) exists in all useful cases since  $r_h$  lies on the ellipse. When this is not true, a negative argument of the radical obtains and indicates that  $r_h$  is less than perigee radius (or greater than apogee). For scaling purposes, the range of values in the denominator of the tangent function is  $(0, 2e)$ . The numerator can lie in the range  $(-e, e)$  as seen from (A-6.1) and (A-6.2).

When  $r_h$  is chosen for reentry,  $t$  in (A-7.1) will be positive for all  $\bar{r}_o$  that move toward  $r_h$  in the direction of  $\bar{v}_o$ , and will become negative when  $\bar{r}_o$  has passed  $r_h$ . This assumes  $0 \leq f_o \leq f$ , where both  $f$  and  $f_o$  are measured positively. However, in general (A-7.1) gives the time of flight from  $\bar{r}_o$  to  $r_h$  and a negative value of  $t$  means that the perigee radius lies on the arc to be traversed and so  $2\pi$  is missing from  $E_o$ . Considered another way, a negative value for  $t$  represents the time of flight from  $\bar{r}_o$  to  $r_h$  in the opposite direction of the velocity and must be added to one orbital period to obtain the time of flight in the desired direction. In either case, to get the proper value for flight time when  $t$  is negative, replace  $t$  by  $t + 2\pi a \sqrt{\frac{a}{\mu}}$

The formulation (A-7) for solving Kepler's equation is somewhat undesirable because (a) it requires evaluating two arctangents, and (b) it may exhibit loss of precision for small values of the difference  $E - E_o$ .

Method II - The formulation of (A-7) can be changed into a more desirable solution that has improved accuracy and speed. The change is accomplished by solving directly for the quantity  $(E - E_o)$ . To enable algebraic manipulation, (A-6.1) is expressed after the manner of (A-6.2).

$$e \sin E_o = \frac{\bar{v}_o \cdot \bar{r}_o}{\sqrt{a\mu}} = \text{sgn}(\dot{r}_o) \sqrt{\frac{r_o}{a} \left(2 - \frac{r_o}{a}\right) - \frac{p}{a}} \equiv Q\left(\frac{r_o}{a}\right) \quad (\text{A-8})$$

Now (A-2) and (A-4) become similar, and are regrouped

$$\tan \frac{E_0}{2} = \frac{Q\left(\frac{r_0}{a}\right)}{1 + e - \frac{r_0}{a}} = \frac{\text{sgn}(\dot{r}_0) \sqrt{e^2 - \left(\frac{r_0}{a} - 1\right)^2}}{e - \left(\frac{r_0}{a} - 1\right)} \quad (\text{A-9.1})$$

$$\tan \frac{E}{2} = \frac{Q\left(\frac{r_h}{a}\right)}{1 + e - \frac{r_h}{a}} = \frac{\text{sgn}(\dot{r}_h) \sqrt{e^2 - \left(\frac{r_h}{a} - 1\right)^2}}{e - \left(\frac{r_h}{a} - 1\right)} \quad (\text{A-9.2})$$

Define the following intermediate sets of variables

$$\begin{aligned} A &\equiv \tan \frac{E_0}{2} = \text{sgn}(\dot{r}_0) \sqrt{\frac{e+c}{e-c}} \\ B &\equiv \tan \frac{E}{2} = \text{sgn}(\dot{r}_h) \sqrt{\frac{e+d}{e-d}} \end{aligned} \quad (\text{A-10})$$

where  $c = r_0/a - 1$  and  $d = r_h/a - 1$ . The definitions of (A-10) are introduced into the trigonometric identity for the tangent of a difference and the numerator rationalized by  $(B + A)$ .

$$\tan \frac{E - E_0}{2} = \frac{\tan \frac{E}{2} - \tan \frac{E_0}{2}}{1 + \tan \frac{E}{2} \tan \frac{E_0}{2}} = \frac{B^2 - A^2}{A(1 + B^2) + B(1 + A^2)}$$

Substitute (A-10), perform the indicated operations, and simplify

$$\tan \frac{E - E_0}{2} = \frac{d - c}{\text{sgn}(\dot{r}_0) \sqrt{e^2 - c^2} + \text{sgn}(\dot{r}_h) \sqrt{e^2 - d^2}}$$

Eliminate  $c$ ,  $d$  and restore as in (A-9)

$$\tan \frac{E - E_0}{2} = \frac{\frac{r_h}{a} - \frac{r_0}{a}}{Q\left(\frac{r_0}{a}\right) + Q\left(\frac{r_h}{a}\right)} \quad (\text{A-11})$$

Take the arctan of (A-11) to get the desired eccentric anomaly difference. As before, negative tangents are considered as having arguments in the range  $(\pi/2, \pi)$ .

The sine term (A-6) in Kepler's equation with the aid of (A-8) now is

$$e(\sin E - \sin E_0) = Q\left(\frac{r_h}{a}\right) - Q\left(\frac{r_0}{a}\right) \quad (\text{A-12})$$

Further simplification is indicated: (A-11) can be put in a more convenient form by observing that the denominator of (A-11) differs from (A-12) by a single minus sign. Rationalize (A-11)

$$\tan \frac{E - E_0}{2} = \frac{Q\left(\frac{r_h}{a}\right) - Q\left(\frac{r_0}{a}\right)}{2 - \frac{r_0}{a} - \frac{r_h}{a}} \quad (\text{A-13})$$

or equivalently

$$\tan \frac{E - E_0}{2} = \frac{e(\sin E - \sin E_0)}{2 - \frac{r_0}{a} - \frac{r_h}{a}} \quad (\text{A-13.1})$$

### Summary - Method II

The time of flight for the elliptic arc in the direction of the velocity  $\bar{v}_0$  from  $\bar{r}_0$  to a radius of specified length  $r_h$  is given by Kepler's equation (A-1)

$$t = a\sqrt{\frac{a}{\mu}} [E - E_0 - e(\sin E - \sin E_0)]$$

where either of the following may be used

$$\begin{aligned} e(\sin E - \sin E_0) &= \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a}} - \text{sgn}(\dot{r}_0) \sqrt{\frac{r_0}{a} \left(2 - \frac{r_0}{a}\right) - \frac{p}{a}} \\ &= \text{sgn}(\dot{r}_h) \sqrt{\frac{r_h}{a} \left(2 - \frac{r_h}{a}\right) - \frac{p}{a}} - \frac{\bar{v}_0 \cdot \bar{r}_0}{\sqrt{a\mu}} \end{aligned} \quad (\text{A-14})$$

and  $E - E_0$  comes from

$$\tan \frac{E - E_0}{2} = \frac{e(\sin E - \sin E_0)}{2 - \frac{r_0}{a} - \frac{r_h}{a}}$$

This requires two parameters  $a$ ,  $p$ .  $\text{Sgn}(\dot{r}_h)$  designates which of the two possible vectors of length  $r_h$  is wanted and is negative for reentry. The radicals have values in the range  $(-e, e)$  and vanish at both apogee and perigee. Negative arguments for the radical containing  $r_h$  signifies that the radius  $r_h$  does not lie on the ellipse. A negative argument for the radical containing  $r_0$  can only arise from numerical effects and indicates that the argument should be zero. Should this situation occur, set the radical equal to zero.

Both the numerator and denominator of the tangent can take on values in the range  $(-2e, 2e)$ . The tangent is indeterminate only in the special case when the radii are apogee and perigee, and corresponds to  $E - E_0 = \pi$ . The arctan will recognize this fact. The usefulness of the tangent equation is not greatly limited by  $e$ , provided  $e \neq 0$  and the range of  $e$  is known in advance for proper scaling.

The selection of one of the two equations for  $e(\sin E - \sin E_0)$  depends on the application. The first one is useful when both signum functions are known in advance. The second is preferred for general application.

As discussed under Summary - Method I, negative values for  $t$  require that  $t$  be replaced by  $t + 2\pi a\sqrt{a/\mu}$ .