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Space Guidance Analysis Memo #26

To: SGA Distribution

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Date: October 23, 1962

Subject: Investigation of Minimum Mid-Course Attitude Maneuvers

This memo presents a method of calculating minimum mid-course attitude maneuvers in the AGC, following the lead of E-1118*. Free use will be made of the following terms defined in E-1118:

$\overline{I}$ - unit vector along spacecraft and roll axis

$\overline{J}$ - unit vector along pitch axis

$\overline{K}$ - unit vector along yaw axis

$\overline{L}$ - unit vector in direction of landmark

$\overline{S}$ - unit vector in direction of star

$\overline{N}$ - unit vector normal to plane of sight ($\overline{N}$ = UNIT($\overline{L} \times \overline{S}$))

$\overline{NOR}$ - unit vector defined by $\overline{NOR} = \overline{L} \times \overline{N}$

$\overline{SDA}$ - unit vector along shaft drive axis

CA - spacecraft half cone angle

$\phi$ - roll angle

$\theta$ - pitch angle

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*Scholten, R. and Philliou, P., Investigation of Midcourse Maneuver Fuel Requirements for Apollo Spacecraft, MIT Inst. Lab. Report E-1118, March 1962

In order to make a star-landmark sighting with the space sextant, the spacecraft must be rotated from its original orientation until the star and landmark are within the field of view of the sextant. The maneuver involves getting the shaft drive axis of the sextant, which is fixed rigidly to the spacecraft, into the plane of sight determined by the vectors to the star and landmark and pointed roughly in the direction of the star landmark combination. Thus, to make a sighting the spacecraft must be oriented so that $\overline{SDA}$ lies in the plane of sight and the angle $\alpha$ between $\overline{L}$ and $\overline{SDA}$ (positive towards $\overline{S}$) lies between the limits $\alpha_L$ and $\alpha_U$ which are determined as follows:

If

$$\beta < \gamma - \delta \qquad \begin{cases} \alpha_U = \beta + \delta \\ \alpha_L = -\delta \end{cases}$$

and, if

$$\beta > \gamma - \delta \qquad \begin{cases} \alpha_U = \gamma \\ \alpha_L = \beta - \delta \end{cases}$$

where $\delta$ and $\gamma$ are the clockwise and counterclockwise limits of the lines of sight about the telescope drive axis and $\beta$ is the magnitude of the angle between $\overline{S}$ and $\overline{L}$. (Presently $\gamma = 50^\circ$ and $\delta = 20^\circ$). To find the maneuver requiring the least fuel, both $\alpha$ and the type of maneuver (e.g. roll-pitch, pitch-roll, roll-yaw), must be varied.

The existence of a roll-only solution - the most desirable from the standpoint of fuel consumption - can be determined as follows. Let $\overline{X}$ be a unit vector representing the direction $\overline{SDA}$ after rolling. Then we must have

$$\overline{I} \bullet \overline{X} = \cos(\pi/2 - CA) = \sin CA \qquad (1)$$

by the geometry of the sextant configuration.

Since $\overline{X}$ must lie in the plane of sight we must have

$$\overline{X} \bullet \overline{N} = 0 \qquad (2)$$

as well. These two simultaneous vector equations in $\overline{X}$ can be solved as follows. Let $\overline{S}_A$ be a unit vector parallel to the projection of $\overline{I}$ on the plane normal to $\overline{N}$. Then

$$\overline{S}_A = \frac{\overline{I} - (\overline{I} \bullet \overline{N})\,\overline{N}}{\sqrt{1 - (\overline{I} \bullet \overline{N})^2}}$$

In addition $\overline{S}_B = \overline{S}_A \times \overline{N}$ is normal to $\overline{S}_A$ and lies in the plane as well. Thus $\overline{X}$ can be written as a linear combination of $\overline{S}_A$ and $\overline{S}_B$ by first finding the projection of $\overline{X}$ on $\overline{S}_A$.

$$\overline{S}_A \bullet \overline{X} = \frac{\overline{I} \bullet \overline{X} - (\overline{I} \bullet \overline{N})\,(\overline{N} \bullet \overline{X})}{\sqrt{1 - (\overline{I} \bullet \overline{N})^2}}$$

$$= \frac{\sin CA - (\overline{I} \bullet \overline{N})\,0}{\sqrt{1 - (\overline{I} \bullet \overline{N})^2}}$$

$$= \frac{\sin CA}{\sqrt{1 - (\overline{I} \bullet \overline{N})^2}} = c$$

Thus there are two soltuions

$$\overline{X} = c\,\overline{S}_A \pm \sqrt{1 - c^2}\;\overline{S}_B$$

provided $|c| < 1$. As Eq (1) represents a cone and (2) a disc, $|c| > 1$ implies no ray of the disc coincides with an element of the cone, and no real solution is available.

However, if $\overline{X}$ does not lie in the region bounded by $\alpha_L$ and $\alpha_U$ it would not be appropriate, since for first examination it was only required that $\overline{X}$ lie in the plane of sight. However, if $\overline{SL}$ and $\overline{SU}$ are unit vectors on the boundary of the region of admissable angles

$$\overline{SL} = \overline{L} \cos \alpha_L + \overline{NOR} \sin \alpha_L$$

$$\overline{SU} = \overline{L} \cos \alpha_U + \overline{NOR} \sin \alpha_U ,$$

$\overline{X}$ can be expressed in terms of $\overline{SL}$ and $\overline{SU}$; i.e. there exist $a_1$, $a_2$ such that

$$\overline{X} = a_1 \overline{SL} + a_2 \overline{SU} .$$

$\overline{X}$ lies in the desired region if and only if both $a_1$ and $a_2$ are positive. Thus for $\overline{X}$ to be a solution we must have both

$$(\overline{SL} \cdot \overline{X}) - (\overline{SU} \cdot \overline{X})(\overline{SL} \cdot \overline{SU}) \geq 0$$

and

$$(\overline{SU} \cdot \overline{X}) - (\overline{SL} \cdot \overline{X})(\overline{SL} \cdot \overline{SU}) \geq 0$$

The correct roll angle $\phi$ can be found as follows. If $\overline{X}_{JK}$ is the unit vector in the direction of the projection of $\overline{X}$ on the JK plane,

then

$$\phi = \cos^{-1} (\overline{X}_{JK} \cdot \overline{K}) \operatorname{sgn} (\overline{X}_{JK} \cdot (-\overline{J})) .$$

Most of the time, however, a roll-only solution will not be possible so "two-stage" maneuvers, such as roll and then pitch which offer much moremobility, must be examined. Note that there are six distinct possibilities to consider as rotational transformations are not commutative: for example, roll-pitch is not generally equivalent to pitch-roll in fuel consumption.

The method of solution for the two-stage maneuver is similar to the method used in roll-only case except that one must first select an admissable direction for $\overline{SDA}$. That is, one must choose $\alpha$ $(\alpha_L \leq \alpha \leq \alpha_U)$ so that $\overline{SDA}_f = \overline{L} \cos \alpha + \overline{NOR} \sin \alpha$ would be the final direction of $\overline{SDA}$. For example, the pitch-roll solution is found as follows. Let $\overline{X}$ be a unit-vector representing the roll-axis after pitching. Certainly $\overline{X} \bullet \overline{J} = 0$. In addition $\overline{X} \bullet \overline{SDA}_f = \sin CA$ must be satisfied if we can roll into $\overline{SDA}_f$ after pitching. As before, the solutions are as follows ($\overline{V}_A$ is the normalized projection of $\overline{SDA}_f$ in the $\overline{I}$, $\overline{K}$ plane etc.):

$$\overline{V}_A = \frac{\overline{SDA}_f - (\overline{SDA}_f \bullet \overline{J})\, \overline{J}}{\sqrt{1 - (\overline{SDA}_f \bullet \overline{J})^2}}$$

$$\overline{V}_B = \overline{V}_A \times \overline{J}$$

$$d = \overline{V}_A \bullet \overline{X} = \frac{\sin CA}{\sqrt{1 - (\overline{SDA}_f \bullet \overline{J})^2}}$$

then

$$\overline{X} = \overline{V}_A\, d \pm \overline{V}_B \sqrt{1 - d^2}$$

provided $| d | < 1$. From this

$$\theta_i = \cos^{-1} (\overline{X}_i \bullet \overline{I}) \operatorname{sgn} (\overline{X}_i \bullet (-\overline{K})) \qquad i = 1, 2$$

and if $\overline{I}'$ and $\overline{K}'$ represent the roll and yaw axes, respectively, after pitching; and $(\overline{SDA}_f)_{JK'}$ is $\overline{SDA}_f$ projected on the $JK'$ plane and normalized, then

$$\phi_i = \cos^{-1} ((\overline{SDA}_f)_{JK}' \cdot \overline{K}_i') \operatorname{sgn} ((\overline{SDA}_f)_{JK}' \cdot (-\overline{J}))$$

$$i = 1, 2$$

Thus, if $|d| < 1$, there are two solutions (coincident if $d = 1$) representing the two intersections of the cone $\overline{X} \cdot \overline{SDA}_f = \sin CA$ and the disc $\overline{J} \cdot \overline{K} = 0$. The fuel requirements for each can be calculated using equation (15) of E-1118, selecting the maneuver requiring the least fuel. Other two-stage maneuvers can be calculated in a similar manner using the same $\overline{SDA}_f$. Then by letting $\alpha$ vary from $\alpha_L$ to $\alpha_U$ (and thus $\overline{SDA}_f$ from $\overline{SL}$ to $\overline{SU}$) a complete set of admissable solutions can be generated, and that requiring the minimum fuel can be selected as the required minimum maneuver.

To gather statistics on fuel consumption and relative frequencies of each type of maneuver, a Monte-Carlo study was made using a hundred random sets of directions uniformly distributed in space. These data were generated using a result from stastistical mechanics that if $\overline{L} = \text{UNIT}(x_1, x_2, x_3)$ where $x_1$, $x_2$, and $x_3$ are normally distributed with the same standard deviation, then the direction of $\overline{L}$ is uniformly distributed in space. Using the MAC "RNDMN" normal random number generator, an $\overline{L}$ was selected. Using the same formula $\overline{S}$ was chosen, and if $\overline{L} \cdot \overline{S} \leq 0$, successive $\overline{S}$'s were chosen until one which satisfied $\overline{L} \cdot \overline{S} > 0$ was found. A hundred such sets constituted the input data.

Using the spacecraft inertias for the translunar vehicle with LEM the following minimum maneuver frequency table was generated:

| Roll-Only | Roll-Pitch | Pitch-Roll | Yaw-Roll | Roll-Yaw |
|---|---|---|---|---|
| 17 cases | 63 cases | 15 cases | 5 cases | 0 cases |

i.e. roll-only proved to be the minimum maneuver 17% of the time, etc. The fuel consumption proved to be a highly variable quantity with experimental mean 25.98 lb-min per maneuver but with standard deviation 16.47. The maximum and minimum were 60.50 and .684 lb-mins of fuel respectively.

Those cases which had neither roll-only nor roll-pitch as minimum solutions were examined to see if the minimum fuel requirement might be approximated by roll-pitch. Tabulation of the minimum roll-pitch solution in these cases - it can be shown that a roll-pitch solution always exists - showed that roll-pitch is quite close to the minimum in any case. Specifically, in the 20 cases to which this applies, the roll-pitch solution was only 3.2% greater on the average with a maximum of 14%. Calling roll-only a special case of roll-pitch yields the result that roll-pitch is the minimum maneuver 80% of the time, and that by following a policy of roll-pitch for each maneuver, only about 2/3% of fuel will be expended above the minimum in the long run.

In conclusion it appears that roll-pitch need be the only maneuver considered, simplifying AGC calculations a great deal. Further studies should be made, however, to determine whether the bias in direction due to landmark sighting has a significant effect on the result.