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Space Guidance Analysis Memo #23-64, (Revision 1)

TO: SGA Distribution  
FROM: William Marscher  
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SUBJECT: Universal Conic Time of Flight Computations

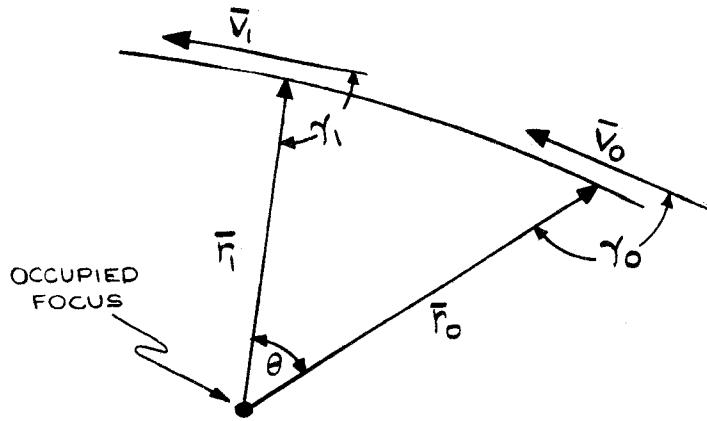
The purpose of this memo is to extend the Herrick Universal formulation of Kepler's time of flight problem, which uses the Battin modified transcendental functions so that five other time of flight problems can also be solved universally. These problems are defined as follows. (See figure for variable definitions.)

- I Lambert's Problem: given  $\bar{r}_0$ ,  $\bar{r}_1$ ,  $t$ ; solve for the conic parameters.
- II Reentry Problem: given  $\bar{r}_0$ ,  $r_1$ ,  $\gamma_1$ ,  $t$ ; solve for the conic parameters.
- III Time-Theta Problem: given  $\bar{r}_0$ ,  $\bar{v}_0$ ,  $\theta$ ; solve for the time of flight.
- IV Time-Pericenter Problem: given  $\bar{r}_0$ ,  $\bar{v}_0$ ; solve for time of flight to pericenter.
- V Time Radius Problem: given  $\bar{r}_0$ ,  $\bar{v}_0$ ; solve for time of flight to  $r_1$ .

An appendix to this memo is attached which covers the following topics:

- (I) Derivation of the above Universal Time of Flight Formula
- (II) Derivation of other equations
- (III) Iteration limits
- (IV) A suggested iteration algorithm

Figure and Variable Definitions



$t$  = time of flight from  $\bar{r}_0$  to  $\bar{r}_1$

$\alpha$  =  $\frac{1}{a}$  where  $a$  = semimajor axis

$p$  = semi-latus rectum

$x$  = Herrick's variable,  $\frac{\Delta E}{\sqrt{\alpha}}$  ellipse,  $\frac{\Delta G}{\sqrt{-\alpha}}$  hyperbola

$S(\text{Aug}) = \frac{1}{3!} - \frac{\text{Aug}}{5!} + \frac{\text{Aug}^2}{7!} - \dots$  (Battin's Transcendental function)

$C(\text{Aug}) = \frac{1}{2!} - \frac{\text{Aug}}{4!} + \frac{\text{Aug}^2}{6!} - \dots$  (Battin's Transcendental function)

$\mu$  = gravitational constant

$r_p$  = pericenter radius

For convenience the following nondimensional variables are defined:

$R = r_0/r_1$

$P = p/r_0$

$X = x/\sqrt{r_0}$

$T = \sqrt{\frac{\mu}{r_0^3}} t$

$A = r_0 \alpha$

$V = \sqrt{\frac{r_0}{\mu}} v_0$

Definition of Those Variables Used in the Appendix Not Previously Defined.

$\Delta E$  = eccentric anomaly difference (ellipse)\*

$\Delta G$  = hyperbolic equivalent to  $\Delta E$ \*

$e$  = eccentricity

$f$  = true anomaly (measured from pericenter)

$v_r$  = radial velocity

$v_c$  = circumferential velocity

$\theta$  = true anomaly difference ( $f_1 - f_0$ )

$h$  = angular momentum

\* See references for definitions.

I. Lambert's Problem (given  $\bar{r}_0, \bar{r}_1, t$ )

$$0 < T < \infty$$

(a) guess  $\cot(\gamma_0)$

$$\text{For } \theta < 180^\circ, \quad c < \cot(\gamma_0) < b$$

$$\text{For } \theta > 180^\circ, \quad -\infty < \cot(\gamma_0) < b$$

where

$$b = \cot\left(\frac{\theta}{2}\right) + \sqrt{\frac{2R}{1 - \cos(\theta)}}$$

$$c = \frac{R - \cos(\theta)}{\sin(\theta)}$$

$$(T = \infty \text{ at } \cot(\gamma_0) = b)$$

NOTE:  $c$  is infinite at  $\theta = 180^\circ$ . Usually a large negative number will suffice for  $c$  in this event.

(b) compute  $P, A$

$$P = \frac{1 - \cos(\theta)}{R - \cos(\theta) + \sin(\theta) \cot(\gamma_0)} \quad (1)$$

$$A = 2 = P \left(1 + \cot^2(\gamma_0)\right) \quad (2)$$

(c) solve for X iteratively

$$\sqrt{PX} \left( \cot \left( \frac{\theta}{2} \right) - \cot (\gamma_0) \right) = \frac{1 - AX^2 S(AX^2)}{C(AX^2)} \quad (3)$$

$$0 < X < \infty$$

NOTE: Practical upper limit for X:

$$A \text{ positive, } X = \frac{2PI}{\sqrt{A}}$$

$$A \text{ negative, } X = \frac{4PI}{\sqrt{|A|}}$$

(d) compute T

$$T = \sqrt{P} \cot (\gamma_0) X^2 C(AX^2) + (1 - A) X^3 S(AX^2) + X \quad (4)$$

(e) If T does not agree with desired T, adjust  $\cot (\gamma_0)$  appropriately. Repeat until agreement is produced.

NOTE: By using the identity

$$PRX^2 C(AX^2) = 1 - \cos (\theta)$$

$C(AX^2)$  can be eliminated from equations (3) and (4), thus requiring the computation of  $S(AX^2)$  only.

II Reentry Problem (given  $\bar{r}_0, r_1, \gamma_1, t$ )

$$0 < T < \infty \text{ for } R > 1$$

(a) guess  $\cot (\gamma_0)$

$$\text{For } R > 1, \quad -\sqrt{\left| 1 + \cot^2(\gamma_1) \right| R^2 - 1} < \cot (\gamma_0) < \sqrt{\left| 1 + \cot^2(\gamma_1) \right| R - 1}$$

$T = \infty$  for  $\cot (\gamma_0)$  at upper limit

For  $R < 1$ , see Appendix.

(b) compute  $\cot(\theta/2)$ , P, A

$$\cot(\theta/2) = \frac{\cot(\gamma_0) + R \cot(\gamma_1)}{(1 - R)} \quad R \neq 1 \quad (5)$$

$$P = \frac{2(R - 1)}{R^2 \left[ 1 + \cot^2(\gamma_1) \right] - \left[ 1 + \cot^2(\gamma_0) \right]} \quad (6)$$

A = use equation (2)

(c) solve for X iteratively using equation (3)

(d) compute T using equation (4)

(e) if T does not agree with desired T, adjust  $\cot(\gamma_0)$  appropriately. Repeat until agreement is produced.

III Time-Theta Problem (given  $\bar{r}_0$ ,  $\bar{v}_0$ ,  $\theta$ )

(a) compute  $\cot(\gamma_0)$ , A, P

$$\cot(\gamma_0) = \frac{\cos(\gamma_0)}{\sqrt{1 - \cos^2(\gamma_0)}} \quad (7)$$

where  $\cos(\gamma_0) = \bar{l}_{r_0} \cdot \bar{l}_{v_0}$

$$A = 2 - V^2 \quad (8)$$

$$P = \frac{2 - A}{1 + \cot^2(\gamma_0)} \quad (9)$$

(b) solve for X iteratively using equation (3)

(c) compute T using equation (4)

IV Time-Pericenter Problem (given  $\bar{r}_0$ ,  $\bar{v}_0$ )

(a) compute  $\cot(\gamma_0)$ , A, P, R,  $\cot(\theta/2)$

$$\cot(\gamma_0) = \text{use equation (7)}$$

$$A = \text{use equation (8)}$$

$$P = \text{use equation (9)}$$

$$R = \frac{1 + \sqrt{1 - PA}}{P} \quad (10)$$

$$\cot(\theta/2) = \text{use equation(5)with } \cot(\gamma_1) = 0$$

(b) solve for X iteratively using equation (3)

(c) compute T using equation (4)

V The Time-Radius Problem (given  $\bar{r}_0, \bar{v}_0, r_1$ )

(a) compute  $\cot(\gamma_0), A, P, \cot(\gamma_1), \cot(\theta/2)$

$$\cot(\gamma_0) = \text{use equation (7)}$$

$$A = \text{use equation (8)}$$

$$P = \text{use equation (9)}$$

$$\cot(\gamma_1) = \pm \sqrt{\frac{2 - A/R}{PR} - 1} \quad (11)$$

$$\cot(\theta/2) = \text{use equation (5)}$$

(b) solve for X iteratively using equation (3)

(c) compute T using equation (4)

NOTE: After P and A are calculated, it might be advisable to check  $r_1$  against  $r_P$  where  $r_P = r_0 P / (1 + \sqrt{1 - PA})$ .

VI Position and Velocity Computations

The following equations are convenient for computing  $\bar{r}$  and  $\bar{v}$  vectors

$$\bar{v}_0 = \sqrt{\frac{\mu P}{r_0}} \left( \cot(\gamma_0) \bar{i}_{r_0} + \bar{i}_{\perp r_0} \right) \quad (12)$$

where

$$\bar{i}_{r_0} = \text{UNIT}(\bar{r}_0)$$

$\bar{i}_{\perp r_0}$  = unit vector normal to  $\bar{r}_0$  in the plane of the conic such that the angle from  $\bar{r}_0$  to  $\bar{i}_{\perp r_0}$  is in the direction of flight.

$$\bar{r}_1 = r_1 \left( \cos(\theta) \bar{i}_{r_0} + \sin(\theta) \bar{i}_{\perp r_0} \right) \quad (13)$$

$$\bar{v}_1 = \sqrt{\frac{\mu R P}{r_1}} \left[ \left( \cot(\gamma_1) \cos(\theta) - \sin(\theta) \right) \bar{i}_{r_0} + \left( \cot(\gamma_1) \sin(\theta) + \cos(\theta) \right) \bar{i}_{\perp r_0} \right] \quad (14)$$

References:

- Herrick, S., Astrodynamical Report, No. 7, July, 1960. Prepared for the AIR Research and Development Command (AFOSR TN-60-773).
- Battin, R. H., Astronautical Guidance, Chapter 2, McGraw-Hill Book Company, Inc., New York, 1964.



## Appendix

### I. Derivation of the Universal Time of Flight Formula

Since there is little written on the derivation of the Herrick Universal Time of Flight Formula, the writer has included this summary as background to this memo.

The key to devising this universal time of flight formula was the recognition by Herrick that  $\Delta E \sqrt{a}$  (defined as  $x$ ) remained finite as an ellipse changed into a parabola and that  $\Delta G \sqrt{a}$  remained finite as a hyperbola changed into a parabola. Herrick evidently reasoned that if the time of flight equation could be expressed in terms of this variable, that a universal formulation might result. Consequently, starting with the standard elliptic form of Kepler's Equation\* in terms of  $\Delta E$ :

$$\sqrt{\frac{\mu}{a^3}} t = \Delta E + \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu a}} \left[ 1 - \cos (\Delta E) \right] - \left( 1 - \frac{r_0}{a} \right) \sin (\Delta E) \quad (1a)$$

and regrouping it as follows:

$$\begin{aligned} \sqrt{\mu} t = a^{3/2} \left[ \Delta E - \sin (\Delta E) \right] + \left( \frac{r_0}{a} \right) a^{3/2} \sin (\Delta E) \\ + \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} a \left[ 1 - \cos (\Delta E) \right] \end{aligned} \quad (2a)$$

and expanding the terms in "[ ]" brackets:

$$a^{3/2} \left[ \Delta E - \sin (\Delta E) \right] = a^{3/2} \left[ \frac{\Delta E^3}{3!} - \frac{\Delta E^5}{5!} + \dots \right] \quad (3a)$$

$$a \left[ 1 - \cos (\Delta E) \right] = a \left[ \frac{\Delta E^2}{2!} - \frac{\Delta E^4}{4!} + \dots \right] \quad (4a)$$

then substituting  $\Delta E = x / \sqrt{a}$  and defining  $U_e = a^{3/2} \left[ \Delta E - \sin (\Delta E) \right]$  and  $C_e = a \left[ 1 - \cos (\Delta E) \right]$  yields

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\* See references for derivation.

$$U_e = \frac{x^3}{3!} - \frac{x^5}{a \cdot 5!} + \frac{x^7}{a^2 \cdot 7!} - \dots$$

$$C_e = \frac{x^2}{2!} - \frac{x^4}{a \cdot 4!} + \frac{x^6}{a^2 \cdot 6!} - \dots$$

These series are finite as the ellipse changes to the parabola ( $a \rightarrow \infty$ ). If  $U_e$  and  $C_e$  are then substituted back into (2a), a time of flight formula valid for the ellipse and parabola results:

$$\sqrt{\mu} t = \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} C_e + \left(1 - \frac{r_0}{a}\right) U_e + r_0 x \quad (5a)$$

The same can be done for the hyperbolic equivalent of Eq. (1a) which is:\*

$$\sqrt{\frac{\mu}{-a^3}} t = -\Delta G + \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{-\mu a}} \left[ \cosh(\Delta G) - 1 \right]$$

$$+ \left[ 1 - \frac{r_0}{a} \right] \sinh(\Delta G) \quad (6a)$$

The resulting series is:

$$U_h = \frac{x^3}{3!} + \frac{x^5}{a \cdot 5!} + \frac{x^7}{a^2 \cdot 7!} + \dots \quad (7a)$$

$$C_h = \frac{x^2}{2!} + \frac{x^4}{a \cdot 4!} + \frac{x^6}{a^2 \cdot 6!} + \dots \quad (8a)$$

If the convention that  $a$ , the semimajor axis, is positive for an ellipse and negative for a hyperbola is adopted, then  $U_e = U_h = U$  and  $C_e = C_h = C$  and a truly universal time of flight equation results:

$$\sqrt{\mu} t = \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} C + \left(1 - \frac{r_0}{a}\right) U + r_0 x \quad (9a)$$

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\* See references for derivation.

The Battin modification to the U and C Series can be obtained by introducing

$$\sqrt{a} = \frac{x}{\Delta E}$$

into Eqs. (3a) and (4a) prior to the expansion:

$$U_e = \frac{x^3 (\Delta E - \sin(\Delta E))}{\Delta E^3} \quad (10a)$$

$$C_e = \frac{x^2 (1 - \cos(\Delta E))}{\Delta E^2} \quad (11a)$$

and expanding the trigonometric functions yields:

$$\frac{\Delta E - \sin(\Delta E)}{\Delta E^3} = \frac{1}{3!} - \frac{\Delta E^2}{5!} + \frac{\Delta E^4}{7!} \dots \quad (12a)$$

$$\frac{1 - \cos(\Delta E)}{\Delta E^2} = \frac{1}{2!} - \frac{\Delta E^2}{4!} + \frac{\Delta E^4}{6!} \dots \quad (13a)$$

The derivation of the hyperbolic case follows a similar pattern. If the convention that "a", the semimajor axis, is negative for a hyperbola is adopted, and the variable x is introduced, the elliptic and hyperbolic series become identical as before.

Battin defined the sin - sinh series as the "S" Series, and the cos - cosh series as the "C" Series. If the argument of the series is chosen as  $\Delta E^2 = \alpha x^2$  and  $\Delta G^2 = \alpha x^2$  then

$$S(\alpha x^2) = \frac{1}{3!} - \frac{\alpha x^2}{5!} + \frac{(\alpha x^2)^2}{7!} + \dots \quad (14a)$$

$$C(\alpha x^2) = \frac{1}{2!} - \frac{\alpha x^2}{4!} + \frac{(\alpha x^2)^2}{6!} - \dots \quad (15a)$$

Note that these series are valid for the Hyperbola only under the assumption that "a" is negative when the argument is hyperbolic.

It is interesting to note the relationship between Herrick's and Battin's series.

$$U = x^3 S(\alpha x^2)$$

$$C = x^2 C(\alpha x^2)$$

The universal time of flight formula in terms of the  $S(\alpha x^2)$  and  $C(\alpha x^2)$  series is now:

$$\sqrt{\mu} t = \frac{\bar{r}_0 \cdot \bar{v}_0}{\mu} x^2 C(\alpha x^2) + (1 - \alpha r_0) x^3 S(\alpha x^2) + r_0 x \quad (16a)$$

It is this form of the universal equation that is used in this memo. Equation (16a) can be expressed in terms of non-dimensional variables as follows:

$$T = \sqrt{P} \cot(\gamma_0) X^2 C(AX^2) + (1 - A) X^3 S(AX^2) + X \quad (17a)$$

where

$$\sqrt{P} \cot(\gamma_0) = \frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} \quad (18a)$$

## II. Derivation of Equations

Equations (1) through (6) will now be derived. The following basic conic equations will be used in the derivations. The reader should refer to the references for their derivations.

$$r = \frac{p}{1 + e \cos(f)} \quad (\text{polar equation of a conic}) \quad (1b)$$

$$h = \sqrt{p\mu} = r v_c = \frac{r v_r}{\cot(\gamma)} = r v \sin(\gamma) \quad (\text{angular momentum equations}) \quad (2b)$$

$$v_r = \sqrt{\frac{\mu}{p}} e \sin(f) \quad (\text{radial velocity}) \quad (3b)$$

\* See next section of Appendix for derivation.

$$v_c = \sqrt{\frac{\mu}{p}} (1 + e \cos (f)) \quad (\text{circumferential velocity}) \quad (4b)$$

$$\frac{v^2}{\mu} = \frac{2}{r} - \frac{1}{a} \quad (\text{energy integral}) \quad (5b)$$

$$p = a(1 - e^2) \quad (\text{definition of eccentricity}) \quad (6b)$$

$$r = a(1 - e \cos (E)) \quad (\text{equivalent of (1b) in terms of the eccentric anomaly (ellipse)}) \quad (7b)$$

$$r = a(e \cosh (G) - 1) \quad ((7b) \text{ for hyperbola}) \quad (8b)$$

$$\cos (f) = \frac{\cos (E) - e}{1 - e \cos (E)} \quad (\text{identity}) \quad (9b)$$

$$\sin (f) = \frac{\sqrt{1 - e^2} \sin (E)}{1 - e \cos (E)} \quad (\text{identity}) \quad (10b)$$

#### A. Derivation of Equation (1)

Equation (1) can be quickly derived by expressing (1b) at position  $\bar{r}_1$  in terms of  $\theta$  and  $f_0$  where  $f_1 = f_0 + \theta$

$$\frac{p}{r_1} = 1 + e \cos (f_0 + \theta) \quad (11b)$$

substituting the identity  $\cos (f_0 + \theta) = \cos (f_0) \cos (\theta) - \sin (f_0) \sin (\theta)$ , and the equation

$$e \sin (f_0) = \frac{p}{r_0} \cot (\gamma_0) \quad (12b)$$

(derived from (2b) and (3b)) and  $\frac{p}{r_0} = 1 + e \cos (f_0)$  into (11b) yields:

$$\frac{p}{r_0} = \frac{1 - \cos (\theta)}{r_0/r_1 - \cos (\theta) + \sin (\theta) \cot (\gamma_0)} \quad (13b)$$

or non-dimensionally

$$P = \frac{1 - \cos(\theta)}{R - \cos(\theta) + \sin \theta \cot(\gamma_0)}$$

### B. Derivation of Equation (2)

Equation (2) is easily obtained by expressing (2b) and (5b) at position  $\bar{r}_0$  and combining to yield

$$\frac{r_0}{a} = 2 - \frac{P}{r_0} (1 + \cot^2(\gamma_0)) \quad (14b)$$

or non-dimensionally

$$A = 2 - P(1 + \cot^2(\gamma_0))$$

### C. Derivation of Equation (3)

Equation 3 can be obtained by starting with the identity

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin(f_1 - f_0)}{1 - \cos(f_1 - f_0)} \quad (15b)$$

and then substituting in the identities below

$$\sin(f_1 - f_0) = \sin(f_1) \cos(f_0) - \cos(f_1) \sin(f_0) \quad (16b)$$

$$\cos(f_1 - f_0) = \cos(f_1) \cos(f_0) + \sin(f_1) \sin(f_0) \quad (17b)$$

followed by the elimination of the true anomaly in favor of the eccentric anomaly using the identities (9b) and (10b) yielding

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin(E_1) \cos(E_0) - \cos(E_1) \sin(E_0) - e \sin(E_1) + e \sin(E_0)}{\sqrt{1 - e^2} \left[ 1 - [\cos(E_1) \cos(E_0) + \sin(E_1) \sin(E_0)] \right]}$$

which, on introducing the eccentric anomaly equivalent of (16b) and (17b), yields

$$\cot \left( \frac{\theta}{2} \right) = \frac{\sin (\Delta E) - e \sin (E_1) + e \sin (E_0)}{\sqrt{1 - e^2} [1 - \cos (\Delta E)]}$$

Introducing the identity

$$\sin (E_1) = \sin (E_0) \cos (\Delta E) + \cos (E_0) \sin (\Delta E)$$

yields

$$\cot \left( \frac{\theta}{2} \right) = \frac{(1 - e \cos (E_0)) \sin (\Delta E)}{\sqrt{1 - e^2} (1 - \cos (\Delta E))} + \frac{e \sin (E_0)}{\sqrt{1 - e^2}}$$

$e \sin (E_0)$  can be eliminated by using (10b), (6b), (7b), and (12b) to produce

$$e \sin (E_0) = \sqrt{1 - e^2} \cot (\gamma_0)$$

(7b), (6b), in addition to the eccentric anomaly, equivalent to (15b) can be introduced to produce

$$\cot \left( \frac{\theta}{2} \right) = \frac{r_0}{\sqrt{pa}} \cot \left( \frac{\Delta E}{2} \right) + \cot (\gamma_0) \quad (18b)$$

which is a relationship between the true and eccentric anomaly differences which contains no ambiguities.

An equation similar to (18b) can be derived for the hyperbolic case:

$$\cot \left( \frac{\theta}{2} \right) = \frac{r_0}{\sqrt{-pa}} \coth \left( \frac{\Delta G}{2} \right) + \cot (\gamma_0) \quad (19b)$$

If Herricks variable  $x$  and the  $S(\alpha x^2)$  and  $C(\alpha x^2)$  series are introduced, a single equation valid for an ellipse, parabola or hyperbola results:

$$\cot \left( \frac{\theta}{2} \right) = \frac{r_0 (1 - \alpha x^2 S(\alpha x^2))}{\sqrt{p} x C(\alpha x^2)} + \cot \gamma_0$$

or nondimensionally:

$$\cot \left( \frac{\theta}{2} \right) = \frac{1 - AX^2 S(AX^2)}{\sqrt{P} XC(AX^2)} + \cot(\gamma_0) \quad (20b)$$

#### D. Derivation of Equation (4)

Equation 4 is derived in Part I. of the Appendix (See Eq. 17a). There remains the derivation of Eq. 18a; which can be derived by expanding the dot product

$$\frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} = \frac{r_0 v_0 \cos(\gamma_0) \sin(\gamma_0)}{\sqrt{\mu} \sin(\gamma_0)}$$

and introducing (2b) to yield

$$\frac{\bar{r}_0 \cdot \bar{v}_0}{\sqrt{\mu}} = \sqrt{p} \cot(\gamma_0)$$

#### E. Derivation of Equation (5)

Equation (5) can be derived from (13b) by writing (13b) for the vector position  $\bar{r}_1$  as follows:

$$\frac{p}{r_1} = \frac{1 - \cos(\theta)}{r_1/r_0 - \cos(\theta) - \sin(\theta) \cot(\gamma_1)} \quad (21b)$$

and eliminating p between (13b) and (21b).

#### F. Derivation of Equation (6)

Equation (6) can be obtained by writing (14b) for vector location  $\bar{r}_1$  as follows:

$$\frac{r_1}{a} = 2 - \frac{p}{r_1} (1 + \cot^2 \gamma_1) \quad (22b)$$

and eliminating  $\frac{p}{r_1}$  between (22b) and (14b).



### III. Iteration Limits

#### Introduction

Iteration of an equation requires that a valid range of the variable being adjusted during the iteration be established.

In all five problems, the iterative solution to Eq. 3 is required. Thus, we must establish a range for  $x$ .

The Lambert and Reentry problems have, in addition, an outer loop of iteration which requires selecting a value of the conic parameter  $\cot(\gamma_0)$ , while the other parameters of the conic are constrained. The selection of  $\cot(\gamma_0)$ , in essence, is the selection of a particular conic section. There are two distinct limits we must observe when selecting a conic section: 1, the conic section must not be a hyperbola about the vacant focus, (characterized by a negative semi-latus rectum "p"); 2, the conic section must not include a flight to infinity and return on a parabola or hyperbola. Should either of these two limits be violated, the results of the computation become meaningless, and generally most iteration algorithms fail to cope with the situation.

#### A. Iteration Range of $x$

From the geometric definition of the eccentric and true anomalies for the ellipse, it is apparent that  $\Delta E = 2\pi$  when  $\theta = 2\pi$  and  $\Delta E = 0$  when  $\theta = 0$ . Since  $x = \Delta E \sqrt{a}$ , we see that  $x_{\max} = 2\pi n \sqrt{a}$  where  $n$  is the number of  $360^\circ$  rotations of the vector  $\bar{r}_1$ . Thus, for the ellipse, the maximum value of  $x$  is infinite and the minimum value of  $x$  is zero.

A convenient geometric interpretation of the limits of  $x$  for a hyperbola is not available, however, Eq. 8b can be used to deduce a limit.

$$r = a(e \cosh(G) - 1) \quad (8b)$$

The range of  $G$  can now be determined to be

$$r \rightarrow \infty \text{ as } G \rightarrow \infty$$

$$r = r_{\text{pericenter}} \text{ when } G = 0$$

Thus, since  $x = \Delta G \sqrt{a}$ , the limits of  $x$  for the hyperbola are the same as for the ellipse.

A practical limit for the elliptic case, which allows one revolution, is obviously  $0 < x < 2\pi\sqrt{a}$ . For the hyperbola, we can use (8b) to establish

$$\Delta G_{\max} \approx \cosh^{-1} \left[ \frac{r_{\max}(e^2 - 1)}{pe} \right] \quad (1c)$$

In practice,  $0 < x < 4\pi$ , is a practical set of limits.

### B. Iteration Range for Lambert's Problem

As previously discussed, we must not allow the conic parameter  $P$  to become negative while adjusting  $\cot(\gamma_0)$ . Equation (1) below can be used to establish this limiting value of  $\cot(\gamma_0)$  which we shall call  $\cot(\gamma_0)_\infty$ .

$$P = \frac{1 - \cos(\theta)}{R - \cos(\theta) + \sin(\theta) \cot(\gamma_0)} \quad (1)$$

It is characteristic of this equation that as  $\theta$  and  $R$  are varied to increase  $P$ ,  $P$  will go to  $+\infty$  and then return as a negative from  $-\infty$ . This occurs when the numerator of (1) is zero or

$$\cot(\gamma_0)_\infty = \frac{\cos(\theta) - R}{\sin(\theta)} \quad (2c)$$

Using Eq. 1 we can reason, if  $\theta < 180^\circ$ ,  $\cot(\gamma_0)$  must be more positive than  $\cot(\gamma_0)_\infty$  to be in a safe region and, if  $\theta > 180^\circ$ ,  $\cot(\gamma_0)$  must be less positive than  $\cot(\gamma_0)_\infty$  to be in a safe region. The limits are then as listed below:

For  $\theta < 180^\circ$

$$\cot(\gamma_0)_\infty < \cot \gamma_0 < +\infty$$

For  $\theta > 180^\circ$

$$-\infty < \cot(\gamma_0) < \cot(\gamma_0)_\infty$$

It is interesting to note that  $\text{arc cot}(\cot(\gamma_0)_\infty)$  is the angle between  $\bar{r}_0$  and  $(\bar{r}_0 - \bar{r}_1)$

A flight through infinity (FTI) can occur only if the conic is parabolic or hyperbolic. Thus, the problem is to avoid becoming parabolic when this leads to a FTI since this is the threshold of trouble. Equation 2 can be combined with Eq. 1 with  $A = 0$  (parabolic condition) to yield the value of  $\cot(\gamma_0)$  when a parabola exists:

$$\cot(\gamma_0) = \cot\left(\frac{\theta}{2}\right) + \sqrt{\frac{2R}{1 - \cos(\theta)}} \quad (3c)$$

There are two solutions. One is an acceptable parabola and the other includes a FTI. We can reason geometrically, using a parabola, that the unacceptable parabola always occurs when the sign of the radical of (3c) is +. This value of  $\cot(\gamma_0)$  we shall call  $\cot(\gamma_0)_p$

$$\cot(\gamma_0)_p = \cot\left(\frac{\theta}{2}\right) + \sqrt{\frac{2R}{1 - \cos(\theta)}} \quad (4c)$$

It is obvious that, independent of  $\theta$ ,  $\cot(\gamma_0)$  must be less positive than  $\cot(\gamma_0)_p$  to be in a safe region. Thus, the limits on  $\cot(\gamma_0)$  are:

$$-\infty < \cot(\gamma_0) < \cot(\gamma_0)_p$$

The problem is to now decide on proper limits among those prescribed by Eqs. (2c) and (4c). Comparing these two sets of limits for  $\theta < 180^\circ$ , we find that

$$\cot(\gamma_0)_\infty < \cot(\gamma_0) < \cot(\gamma_0)_p$$

We can also reason that  $T = 0$  at  $\cot(\gamma_0)_\infty$  and  $T = \infty$  at  $\cot(\gamma_0)_p$

Again, comparing the two sets of limits for  $\theta > 180^\circ$ , we see that the parabolic limit  $\cot(\gamma_0)_p$  is less than  $\cot(\gamma_0)_\infty$ . There then is no lower limit for  $\cot(\gamma_0)$  other than that value which creates a flight down  $\bar{r}_0$  and up  $\bar{r}_1$ . The limits therefore are:

$$-\infty < \cot(\gamma_0) < \cot(\gamma_0)_p$$

again  $T = \infty$  at  $\cot(\gamma_0)_p$  and  $T = 0$  at  $\cot(\gamma_0) = -\infty$ .

### C. Iteration Range for the Reentry Problem

As previously discussed, we must avoid a negative P when adjusting  $\cot(\gamma_0)$  while iterating the reentry problem. With the constraints imposed by the reentry problem, Eq. 6 can be used to establish these limits.

$$P = \frac{2(R - 1)}{R^2[1 + \cot^2(\gamma_1)] - [1 + \cot^2(\gamma_0)]} \quad (6)$$

It is obvious that for  $R > 1$ , the denominator must not be negative; and for  $R < 1$ , the denominator must not be positive. The threshold in either case is when the denominator is zero. Setting the denominator to zero yields:

$$\cot(\gamma_0)_n = \pm \sqrt{[1 + \cot^2(\gamma_1)] R^2 - 1}$$

Let

$$\cot(\gamma_0)_\infty = + \cot(\gamma_0)_n \quad (5c)$$

Using (6), it is obvious that

For  $R > 1$

$$- \cot(\gamma_0)_\infty < \cot(\gamma_0) < \cot(\gamma_0)_\infty$$

For  $R < 1$ , we observe, using Eq. 6, that  $\cot^2(\gamma_0)$  must increase from  $\cot^2(\gamma_0)_\infty$  if  $\cot(\gamma_0)$  is to stay in a safe region. This being the case, we realize there are two regions of safeness. We also observe that the radicand is about to become negative when  $R = \sin(\gamma_1)$ . Using Eq. 6, we realize that P can no longer become negative when  $R < \sin(\gamma_1)$ . Thus, the limits are:

For  $R < 1$

For  $R > \sin(\gamma_1)$

$$\cot(\gamma_0)_\infty < \cot(\gamma_0) < + \infty$$

$$-\infty < \cot(\gamma_0) < -\cot(\gamma_0)_\infty$$

For  $R < \sin(\gamma_1)$

$$0 < \cot(\gamma_0) < +\infty$$

$$-\infty < \cot(\gamma_0) < 0$$

Now turning our attention to the flying through infinity (FTI) limits. These limits are conveniently studied by eliminating P between Eqs. (2) and (6), setting  $A = 0$ , and solving for  $\cot(\gamma_0)$ , denoted by  $\cot(\gamma_0)_p$

$$\cot(\gamma_0)_p = \pm \sqrt{(1 + \cot^2(\gamma_1)R - 1)} \quad (6c)$$

Using a parabola, we can reason geometrically that for  $R > 1$ , the  $+\cot(\gamma_0)_p$  is the unacceptable parabola and the safe region is more negative than this and for  $R < 1$ , there are a very complex set of limits. For  $R < \sin^2(\gamma_1)$ , the conic section cannot become parabolic. For  $\cot(\gamma_1) > 0$ ,  $\cot(\gamma_0)_p$  yields two safe parabolas. For  $\cot(\gamma_1) < 0$ ,  $\cot(\gamma_0)_p$  yields two unsafe parabolas and  $\cot^2(\gamma_0)$  must increase from this value.

The limits to avoid a FTI are:

$R < 1$

$$\cot(\gamma_1) < 0 \text{ and } R > \sin^2(\gamma_1)$$

$$\cot(\gamma_0)_p < \cot(\gamma_0) < +\infty$$

$$-\infty < \cot(\gamma_0) < -\cot(\gamma_0)_p$$

Otherwise

$$0 < \cot(\gamma_0) < +\infty$$

$$-\infty < \cot(\gamma_0) < 0$$

$R > 1$

$$-\infty < \cot(\gamma_0) < \cot(\gamma_0)_p$$

Comparing the two sets of limits, (infinite P and FTI), we see that the limits for  $R < 1$  are exceedingly complex and the limits for  $R > 1$ , recognizing that  $\cot(\gamma_0)_p < \cot(\gamma_0)_\infty$ , are as follows:

$R > 1$

$$-\cot(\gamma_0)_\infty < \cot(\gamma_0) < \cot(\gamma_0)_p$$

$$\cot(\gamma_0) = -\cot(\gamma_0)_\infty, \quad T = 0$$

$$\cot(\gamma_0) = \cot(\gamma_0)_p, \quad T = \infty$$

#### IV. A Suggested Iteration Algorithm

The iteration of the equations in this memo can be carried out with any of a number of algorithms. One very simple algorithm which the author has used with success is as follows:

- (A) Establish the maximum and minimum values of the adjusted variable. (In the case of  $\cot(\gamma_0)$ , the maximum and minimum must be decremented and incremented, respectively, by a small amount to place them in a safe region.)
- (B) Start the iteration at the minimum value.
- (C) Compute the resulting error.
- (D) Test the error to see if it is satisfactorily small. If so, the iteration is complete. If not, continue.
- (E) If the sign of the error has not changed from the sign which resulted when (B) was computed, reset the minimum value of the adjusted variable to the present value. If it has changed, reset the maximum value to the present value.
- (F) Compute a new adjusted variable as follows:

$$\text{new adjusted variable} = \text{min. value} + \left( \frac{\text{max. value} - \text{min. value}}{2} \right)$$

- (G) Return to (C) and repeat.