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Space Guidance Analysis Memo #22

To: SGA Distribution

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Subject: Calculation of the Roots of u C (u) = A

Let

$$C(u) = \frac{1 - \cos(\sqrt{u})}{u}$$

for $u \geq 0$. Then $0 \leq u\,C(u) \leq 2$. Let A be a number between 0 and 2 and consider the roots of the equation

$$u\,C(u) - A = 0$$

lying between 0 and π^2. If u_0 is a root we have

$$\cos(\sqrt{u_0}) = 1 - A$$

Remark (a) $u\,C(u) - A = 0$ has exactly one root between 0 and π^2.

The feasibility of calculating this root using the recursion formula

$$u_{n+1} = \frac{A}{C(u_n)}$$

will now be investigated.

Remark (b) Let v denote the root and let u_0 be the initial guess at the root. If $0 \leq u_0 < v$, then $u_{n+1} > u_n$ for every n and $u_n \rightarrow v$ as $n \rightarrow \infty$. Similarly, if $v < u_0 \leq \pi^2$, then $u_{n+1} < u_n$ for each n and $u_n \rightarrow v$ as $n \rightarrow \infty$.

Proof of Remark (b). We will start with two lemmas.

Lemma (1) $u\,C(u)$ is (strictly) monotone increasing and $C(u)$ is (strictly) monotone decreasing for $0 \leq u \leq \pi^2$. (Strict monotonicity means that if $u_1 > u_2$ then $C(u_1) > C(u_2)$ and not $C(u_1) \geq C(u_2)$; that is $C(u_1) \neq C(u_2)$.)

Proof of Lemma (1).

$$u\,C(u) = 1 - \cos(\sqrt{u}).$$

Since $\cos(\sqrt{u})$ is monotone decreasing for $0 \le u \le \pi^2$, it follows that $u\,C(u)$ is monotone increasing.

Let

$$f(x) = \sin x - x\cos x$$

then

$$f(0) = 0 \qquad (1)$$

$$f'(x) = x\sin x$$

and

$$f'(x) \ge 0 \qquad (2)$$

for $0 \le x \le \pi$. From (1) and (2) it follows that

$$f(x) \ge 0 \qquad (3)$$

for $0 \le x \le \pi$.

Let

$$g(x) = \frac{\sin x}{x}$$

then

$$g(0) = 1 \qquad (4)$$

$$g'(x) = \frac{x\cos x - \sin x}{x^2} = -\frac{1}{x^2}\,f(x).$$

Therefore

$$g'(x) \le 0 \qquad (5)$$

for $0 \le x \le \pi$.

Furthermore

$$g(\pi) = 0 \qquad (6)$$

Therefore, from (4), (5) and (6) it follows that $g(x)$ is monotone decreasing for $0 \le x \le \pi$ and

$$0 \le g(x) \le 1$$

for x in this range.

Finally,

$$\frac{1}{2}\left\{g\left(\frac{1}{2}\sqrt{u}\right)\right\}^2 = \frac{2\sin^2\left(\frac{1}{2}\sqrt{u}\right)}{u}$$

$$= \frac{1 - \cos(\sqrt{u})}{u}$$

$$= C(u).$$

Since g (x) is monotone decreasing and positive, it follows that C (u) is monotone decreasing.

Lemma (2) If $0 \leq u_n < v$, then $u_n < u_{n+1} < v$.

Proof of Lemma (2). By Lemma (1) C (u) is monotone decreasing and

$$C(v) < C(u_n).$$

Therefore

$$A = v\,C(v) < v\,C(u_n)$$

or

$$u_{n+1} = \frac{A}{C(u_n)} < v.$$

Since (Lemma (1)) u C (u) is monotone increasing

$$u_n\,C(u_n) < v\,C(v) = A$$

and therefore

$$u_n < \frac{A}{C(u_n)} = u_{n+1}.$$

Proof of remark continued. If $0 \leq u_0 < v$, it follows by Lemma (2) and induction that the u_n's form an increasing sequence bounded above by v. Therefore, the u_n's approach a limit u_∞. Taking limits on both sides of the recursion formula

$$u_{n+1} = \frac{A}{C(u_n)}$$

we have

$$u_\infty = \frac{A}{C(u_\infty)}$$

or

$$u_\infty\,C(u_\infty) = A$$

Furthermore, since

$$0 \leq u_n < \pi^2$$

we have

$$0 \leq u_\infty \leq \pi^2.$$

Since v is the only root of $u\,C(u) - A = 0$ between 0 and π^2, $u_\infty = v$. This completes the proof of the first half of Remark (b). The proof of the second half follows the same lines.

Now consider the roots of the equation

$$u\,C(u) - A = 0 \qquad (7)$$

when A is negative. We are interested in the roots lying between $-\infty$ and π^2. $u\,C(u)$ is positive when u lies between 0 and π^2 so any possible roots are negative. For u negative

$$u\,C(u) = -\left\{\cosh(\sqrt{-u}) - 1\right\} .$$

Let $B = -A$ and $w = -u$. Then equation (7) becomes

$$w\,C(-w) = B$$

or

$$\cosh(\sqrt{w}) = B + 1$$

<u>Remark (c)</u> For $A < 0$, equation (7) has exactly one root and this root is negative.

The feasibility of calculating the root of equation (7) using the recursion formula

$$u_{n+1} = \frac{A}{C(u_n)}$$

for negative values of A will now be investigated. In order to have positive variables, the recursion formula will be rewritten in the form

$$w_{n+1} = \frac{B}{C(-w_n)} .$$

Also let

$$z = -v$$

where v is the root of equation (7).

<u>Remark (d)</u> If $w_n < z$, then $w_{n+1} > z$ and if $w_n > z$, then $w_{n+1} < z$. Thus the successive w_n's oscillate about the root.

Proof of Remark (d). Using the power series expansion for the hyperbolic cosine we have

$$C(-w) = \frac{\left(1 + \frac{w}{2!} + \frac{w^2}{4!} + \ldots\right) - 1}{w}$$

$$C(-w) = \frac{1}{2!} + \frac{w}{4!} + \frac{w^2}{6!} + \ldots$$

Since w^n is strictly increasing, C (-w) is the sum of strictly increasing functions and is therefore strictly monotone increasing for positive w. Let

$$F(w) = \frac{B}{C(-w)} .$$

Since C (-w) is strictly monotone increasing, F (w) is strictly monotone decreasing.

Furthermore

$$w_{n+1} = F(w_n)$$

and

$$z = F(z) .$$

Assume that $w_n < z$. Then, since F (w) is monotone decreasing,

$$w_{n+1} = F(w_n) > F(z) = z$$

or

$$w_{n+1} > z .$$

Similarly, if $w_n > z$,

$$w_{n+1} = F(w_n) < F(z) = z$$

or

$$w_{n+1} < z .$$

Unlike the case when $A > 0$, w_n does not converge to z if -A is too large. Let x_0 be the positive nonzero root of the equation

$$x = 2 \tanh x$$

let

$$z_0 = 4\,x_0^{\,2}$$

and let

$$B_0 = z_0\, C\,(-z_0)\ .$$

Remark (e) If $0 < B < B_0$ and the initial guess w_0 is sufficiently good, then $w_n \to z$ as $n \to \infty$. If $B > B_0$, then the u_n's oscillate and do not approach a limit as $n \to \infty$.

(To three decimal places:

$$x_0 = 1.91$$

$$z_0 = 14.6$$

$$B_0 = 21.8$$

For a given value of B it is not easy to tell whether an initial guess w_o is sufficiently good to insure convergence. By using the methods of the following proof and the mean value theorem it can be shown that if one makes the initial guess $w_0 = z_0$ and B is less than B_0, then $w_n \to z$.)

Proof of Remark (e). Again let

$$F\,(w) = \frac{B}{C(-w)}$$

Then

$$F\,(w) = \frac{B\,w}{\cosh\,(\sqrt{w})\,-1}\ .$$

Let

$$G\,(w) = F'\,(w) = B\left[\cosh\,(\sqrt{w}) - 1 - \frac{\sqrt{w}}{2}\sinh\sqrt{w}\right]\Big/\left[\cosh\,(\sqrt{w}) - 1\right]^2$$

Since

$$B = \cosh\,(\sqrt{z}) - 1$$

we have

$$G(z) = -\frac{\sqrt{z}}{2} \frac{\sinh(\sqrt{z})}{\cosh(\sqrt{z}) - 1} + 1 \qquad (8)$$

It is necessary to determine the values of z for which $G(z) = -1$. By (8) this is the same as solving the equation

$$\frac{\sqrt{z}}{2} \frac{\sinh(\sqrt{z})}{\cosh(\sqrt{z}) - 1} = 2 \qquad (9)$$

If z_0 is a root of (9) we have

$$\sqrt{z_0} \sinh(\sqrt{z_0}) = 8 (\cosh(\sqrt{z_0}) - 1)$$

Using the identities

$$\cosh a - 1 = 2 \sinh^2 (\tfrac{a}{2})$$

and

$$\sinh a = 2 \sinh(\tfrac{a}{2}) \cosh(\tfrac{a}{2})$$

we have

$$\sqrt{z_0} = 4 \tanh \left(\frac{\sqrt{z_0}}{2}\right) . \qquad (10)$$

Letting $z_0 = 4x^2$, (10) becomes

$$x = 2 \tanh x \qquad (11)$$

Since $G(0) = 0$ and $G(z) \to -\infty$ as $z \to \infty$, $G(z) = -1$ for at least one value of z. By considering the graphs of $y = x$ and $y = 2 \tanh x$ one can see that (11) has one positive root x_0 as well as the root $x = 0$. The root $x = 0$ can be discarded since $G(0) \neq -1$. Since $\frac{\sqrt{z_0}}{2}$ satisfies (11) for every root z_0 of $G(z) = -1$ and x_0 is the only acceptable root of (11), there must be exactly one root of $G(z) = -1$ and

$$z_0 = 4 x_0^2 .$$

As $G(z) = -1$ has only one root, it follows that

$$G(z) > -1$$

if $z < z_0$ and

$$G(z) < -1$$

if $z > z_0$.

It will now be shown that, if $B > B_0$, then the w_n's do not converge as $n \to \infty$. Suppose that $w_n \to w_\infty$ as $n \to \infty$.

$$w_{n+1} = \frac{B}{C(-w_n)} .$$

Taking limits

$$w_\infty = \frac{B}{C(-w_\infty)} .$$

or

$$w_\infty C(-w_\infty) = B .$$

Therefore $w_\infty = z$. Moreover

$$w_{n+1} = F(w_n)$$

and

$$z = F(z) .$$

Combining the last two equations we have

$$W_{n+1} - z = F(w_n) - F(z) .$$

Thus

$$\left| \frac{w_{n+1} - z}{w_n - z} \right| = \left| \frac{F(w_n) - F(z)}{w_n - z} \right| . \qquad (12)$$

As $n \to \infty$

$$\frac{F(w_n) - F(z)}{w_n - z} \to F'(z) = G(z) .$$

Since $z > z_0$, $|G(z)| > 1$. Choose θ so that $1 < \theta < |G(z)|$. Then for n greater than or equal to some number N,

$$\left| \frac{F(w_n) - F(z)}{w_n - z} \right| > \theta .$$

By (12) it follows that

$$\frac{|w_{n+1} - z|}{|w_n - z|} > \theta$$

for $n \geq N$. Therefore

$$|w_n - z| > |w_N - z| \theta^{n-N}$$

for $n \geq N$ and hence

$$|w_n - z| \to \infty$$

as $n \to \infty$, since $\theta > 1$. This contradicts the assumption that $w_n \to z$ as $n \to \infty$. The proof that $w_n \to z$ as $n \to \infty$ if $B < B_0$ is similar.