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SGA Memo #14-65

TO:	SGA Distribution
FROM:	Edward M. Copps, Jr.
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SUBJECT:	Time to go and Delta-v

This memo documents the development of equations suitable for accurately estimating the delta-v and the time of burn for a powered flight maneuver before ignition. The equation includes second order effects associated with gravity losses.

The delta-v estimate is suitable for use on the Δv remaining meter on the main display console, after modification for uncertainties.

The development assumes a thrust versus time relationship that yields slightly longer time of burn than a constant thrust engine. This effect does not carry over into the delta-v estimate.

The equations are available in a MAC program at MIT named TGO220EC which may be called as follows:

CALL TGO220EC, $\overline{V}G$, $\overline{B}$, TAU, A_T

the variables are

$\overline{V}G$	-	velocity to be gained at ignition
$\overline{B}$	-	the B vector at ignition (see SGA Memo 13-64 rev. 1)
TAU	-	GI_{sp}/a_T, the thrust acceleration at ignition divided by the derivative of thrust acceleration at ignition
A_T	-	thrust acceleration of ignition

The subroutine will print two numbers; time of burn and delta-v, in units consistent with the input units. The main program can continue with a statement

RESUME V1, V2, V3

where V1 will be time of burn, V2 will be delta-v, and V3 will be zero if the data is usable. In the words of Jim Miller "This program is not known not to work".

An example of the accuracy of the estimate is taken from one of the burns on flight 204:

	time of burn	delta-v
estimate	23.565	631.75
actual	23.552	631.83

Derivation

The diagram of accelerations acting on the vehicle during powered flight is shown in Fig. . This diagram corresponds to the steer law which places $-\dot{\overline{V}}_g$ along $\overline{V}_g$. The $\overline{b}$ vector is proportional to the $\overline{V}_g$ vector by the matrix equation

$$\overline{b} = C^* \overline{V}_g$$

we presume that C^* is constant. Since C^* is constant and the steer law renders $\overline{V}_g$ irrotational, $\overline{b}$ is constant in direction in space, and proportional to $\overline{V}_g$. We notate this as

$$b = c^* v_g \quad . \tag{1}$$

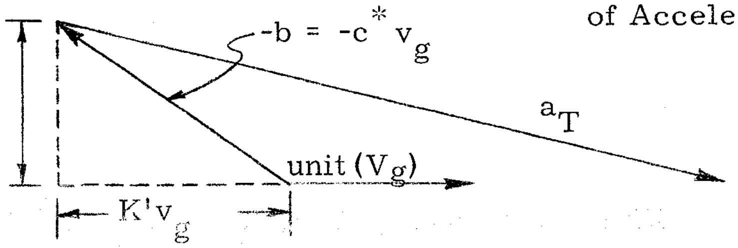

Figure 1 Diagram of Acceleration

The constants K'' and K' are determined from the conditions at ignition as follows:

$$K' = \overline{b} \cdot \overline{V}_g / (\overline{V}_g \cdot \overline{V}_g) \tag{2}$$

and

$$K'' = \sqrt{\overline{b} \cdot \overline{b} / (\overline{V}_g \cdot \overline{V}_g) - K'^2} \quad . \tag{3}$$

The magnitude of $\dot{V}_g$ is obtained by reference to Fig. 1 as

$$\dot{v}_g = -\sqrt{a_T^{\,2} - K''^2 V_g^{\,2}} + K' V_g \quad . \tag{4}$$

This equation is simplified by binomial expansion of the square root term to the time varying non-linear equation

$$\dot{v}_g = - a_T(t) + \frac{K''^2}{2\, a_T(t)} v_g^{\,2} + K' v_g \,, \tag{5}$$

which is recognized (with Ray Morth's help) as Ricatti's equation. A standard form of this equation is

$$\frac{dy}{dt} + P(t)\, y + Q(t)\, y^2 = R(t). \tag{6}$$

which is reduced to the linear, second order, time varying equation

$$\frac{d^2u}{dt^2} + \left(P(t) - \frac{\dot{Q}(t)}{Q(t)} \right) \frac{du}{dt} - R(t)\, Q(t)\, u = 0 \,, \tag{7}$$

by the change of variable

$$Q(t)\, y = \frac{du/dt}{u} \quad . \tag{8}$$

By this method, equation 5 becomes

$$\frac{d^2u}{dt^2} + \left(- K' + \frac{\dot{a}_T}{a_T} \right) \frac{du}{dt} - \frac{K''^2}{2} u = 0 \,, \tag{9}$$

with u defined by

$$- K''^2\, v_g / 2\, a_T = \frac{du/dt}{u} \quad . \tag{10}$$

the term $\frac{\dot{a}_T}{a_T}$ is the only time varying coefficient in this equation. The solution is particularly simple if the acceleration is constant. ($\dot{a}_T = 0$). Moreover, it is interesting to note that because the time varying term is specifically $\dot{a}_T/a_T$, we can retain a constant coefficient by observing that

$$\dot{a}_T/a_T = \text{constant} = D \tag{11}$$

implies a time function

$$a_T(t) = a_o\, e^{Dt} = a_o\,(1 + Dt + \frac{D^2 t^2}{2} + \ldots) \;. \tag{12}$$

We compare this to the expansion that results from the acceleration of a constant thrust rocket engine

$$a_T(t) = a_o\,(1 + \frac{t}{\tau} + \frac{t^2}{\tau^2} + \ldots) \tag{13}$$

where $\tau = m/\dot{m}$. ($\dot{m}$ is a positive number) (14)

Comparison of terms indicates that the substitution

$$\frac{1}{\tau} = \dot{a}_T/a_T \tag{15}$$

in Eq. will yield a solution for an acceleration profile that matches that expected from a rocket to the first order in time, yet which obtains constant coefficients in the equation to be solved.

Returning to Eq. 9 , we simplify notation and rewrite the equation

$$\ddot{u} + a_1\,\dot{u} + a_o\, u = 0 \;. \tag{16}$$

The solution of this equation can be written

$$u = c_1 e^{\lambda_1 t} + c_2\, e^{\lambda_2 t} \tag{17}$$

where

$$\lambda_1 = -a_1/2 + \frac{1}{2}(a_1^{\,2} - 4a_o)^{1/2}$$

$$\lambda_2 = -a_1/2 - \frac{1}{2}(a_1^{\,2} - 4a_o)^{1/2} \tag{18}$$

the derivative of this solution is

$$\dot{u}(t) = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \quad . \tag{19}$$

Returning to the original variables:

$$\frac{-K''^2 v_g}{2 a_T} = \frac{\lambda_1 e^{\lambda_1 t} + C \lambda_2 e^{\lambda_2 t}}{e^{\lambda_1 t} + C e^{\lambda_2 t}} \quad . \tag{20}$$

The remaining constant of integration is obtained by specifying that $t = 0$ at ignition. Let

$$\omega = \frac{K''^{\,2} v_{go}}{2 a_{To}} , \tag{21}$$

then

$$C = -\frac{\lambda_1 + \omega}{\lambda_2 + \omega} \quad .$$

The time of burn is obtained by letting $v_g = 0$;

$$0 = \frac{\lambda_1 e^{\lambda_1 T} + C \lambda_2 e^{\lambda_2 T}}{e^{\lambda_1 T} + C e^{\lambda_2 T}} \tag{22}$$

which has a solution at

$$\lambda_1 e^{\lambda_1 T} = \frac{\lambda_1 + \omega}{\lambda_2 + \omega} \lambda_2 e^{\lambda_2 T} \quad . \tag{23}$$

yielding finally

$$T = \log \left(\frac{\lambda_2 (\lambda_1 + \omega)}{\lambda_1 (\lambda_2 + \omega)} \right) \Big/ (\lambda_1 - \lambda_2) \tag{24}$$

where

$$\omega = \frac{K''^2 v_{g\,o}}{2\, a_{T\,o}}$$

$$\lambda_1 = -1/2\tau + K'/2 + \frac{1}{2} \left(K'^2 + 2K''^2 - \frac{1}{\tau} \left(-\frac{1}{\tau} + 2K'\right)\right)^{1/2}$$

$$\lambda_2 = -1/2\tau + K'/2 - \frac{1}{2} \left(K'^2 + 2K''^2 - \frac{1}{\tau} \left(-\frac{1}{\tau} + 2K'\right)\right)^{1/2}$$

and K', and K'' and τ are defined by Eqs. 22 and 15.