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TO: SGA Distribution
FROM: Edward Womble
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SUBJECT: A Recursive CG Correction Scheme

## Introduction

At the initiation of thrust, a disturbing moment on the vehicle can be produced by an initial misalignment of the thrust vector relative to the vehicle CG. This misalignment can be caused by:

(1) An uncertainty in the information, on the CG location, used by the astronaut to pre-align the engine gimbal servos.

(2) A mechanical offset in the positioning of the engine nozzle by the pitch and yaw gimbal servos.

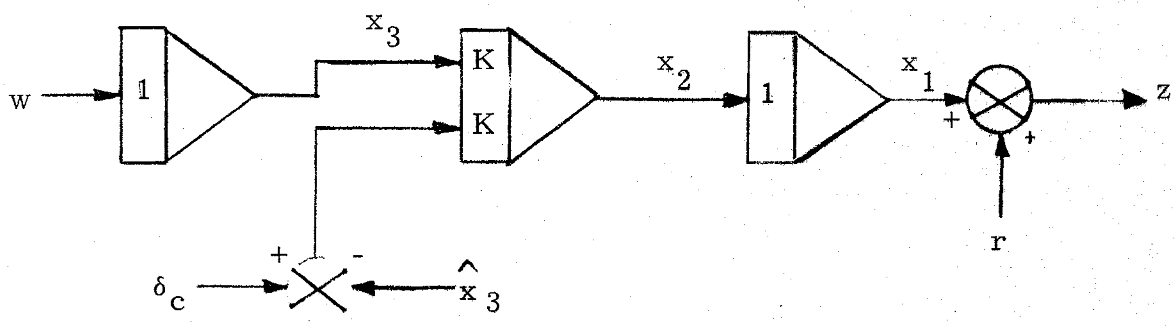
(3) A misalignment of the thrust vector relative to the nozzle center-line.

The necessity for stabilizing low-frequency slosh and bending modes of the CSM - LM vehicle places some severe limitations on the gain and bandwidth of the CSM - LM digital autopilot filter. As a result of these limitations, the use of the autopilot feedback loop to generate a signal to compensate for thrust misalignment would result in the buildup of excessive attitude errors. Therefore, an auxilliary external correction scheme is needed to augment autopilot action in providing this misalignment correction. A number of such schemes have been proposed and are being investigated.

A method for recursively canceling the effects of the disturbing moment is presented here. A correction signal is applied to each engine gimbal servo immediately after the first attitude measurement is received from the IMU, and is updated with each subsequent measurement.

## Statement of Problem

The model of the vehicle used for this derivation is shown in Fig. 1.




In Fig. 1:

(1) w (t) is a driving signal characterized by unbiased Gaussian noise with the known covariance $\sigma_w^2$.

(2) r (t) is a measurement error characterized by unbiased Gaussian noise with the known covariance $\sigma_r^2$.

(3) z is the IMU measurement.

(4) The state is defined to be

$$\underline{x} = [\theta_{RB}, \dot{\theta}_{RB}, \delta_b]^T,$$

where $\theta_{RB}$ and $\dot{\theta}_{RB}$ are the vehicles rigid body angular position and angular velocity, and $\delta_b$ is the disturbing moment.

(5) The initial value of the covariance of the state estimate, $P(\hat{x}(0))$, is known.

(6) $\hat{x}_3$ is the previous best estimate of $x_3$.

(7) $\delta_c$ is the commanded engine position.

The difference equation corresponding to the state diagram in Fig. 1 is:

$$\underline{x}(n+1) = \phi \, \underline{x}(n) + \Gamma \, \underline{u}(n), \qquad (1)$$

where

$$\phi = \begin{bmatrix} 1 & \Delta T & k\frac{\Delta T^2}{2} \\ 0 & 1 & k\,\Delta T \\ 0 & 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} k\frac{\Delta T^2}{2} & \frac{\Delta T^3}{6} \\ k\,\Delta T & \frac{\Delta T^2}{2} \\ 0 & \Delta T \end{bmatrix}$$

$$\underline{u}(n) = [\delta_c(n) - \hat{x}_3(n), \; w(n)]^T.$$

The best estimate of the state at the n + 1 sample, prior to the n + 1 measurement, is

$$\overline{\underline{x}}(n+1) = \phi \, \hat{\underline{x}}(n) + \Gamma \, \overline{\underline{u}}(n), \qquad (2)$$

where

$$\overline{\underline{u}}(n) = [\delta_c(n) - \hat{x}_3(n), \quad 0]^T$$

and

$$Q(n) = \overline{\overline{\underline{u}}(n)\,\overline{\underline{u}}(n)^T} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_w^2 \end{bmatrix}.$$

The problem is now to determine the value of $\delta_b$ from the noisy measurement z.

## Derivation

Subtracting (1) from (2) yields:

$$\underline{e}\,(n+1) = \overline{\underline{x}}\,(n+1) - \underline{x}\,(n+1) = \phi\left[\hat{\underline{x}}\,(n) - \underline{x}\,(n)\right] + \Gamma\left[\overline{\underline{u}}\,(n) - \underline{u}\,(n)\right].$$

Therefore, the estimation covariance at the n + 1 sample, prior to the n + 1 measurement, is

$$M\,(n+1) = \overline{\underline{e}\,(n+1)\,\underline{e}\,(n+1)^T} = \phi\,P\,(n)\,\phi^T + \Gamma Q\,(n)\,\Gamma^T$$

$$+\ \phi\ \overline{[\hat{\underline{x}}\,(n) - \underline{x}\,(n)]\ [\overline{\underline{u}}\,(n) - \underline{u}\,(n)]^T}\ \Gamma^T$$

$$+\ \Gamma^T\ \overline{[\overline{\underline{u}}\,(n) - \underline{u}\,(n)]\ [\underline{x}\,(n) - \underline{x}\,(n)]}\ \phi$$

However,

$$\overline{\underline{u}}\,(n) - \underline{u}\,(n) = \begin{bmatrix} [\delta_c\,(n) - \hat{x}_3\,(n)] - [\delta_c\,(n) - \hat{x}_3\,(n)] \\ \\ 0 - w\,(n) \end{bmatrix} = -\begin{bmatrix} 0 \\ \\ w\,(n) \end{bmatrix}$$

, $\overline{w\,(n)} = 0$, and $w\,(n)$ and $[\hat{\underline{x}}\,(n) - \underline{x}\,(n)]$ are independent, therefore,

$$M\,(n+1) = \phi\,P\,(n)\,\phi^T + \Gamma Q\,(n)\,\Gamma^T. \qquad (3)$$

If a signal with a gaussian distribution is passed through a linear system, the output of the linear system has a gaussian distribution. Therefore, the distribution function for $\underline{x}\,(n)$ is:

$$f_1 [\underline{x}(n)] = \frac{1}{(2\pi)^{3/2} \, |M(n)|^{1/2}} e^{-1/2 [\underline{x}(n) - \bar{\underline{x}}(n)] M^{-1}(n) [\underline{x}(n) - \bar{\underline{x}}(n)]^T} \tag{4}$$

The probability density function for the measurement noise is

$$f_2 [r(n)] = \frac{1}{(2\pi)^{1/2} \sigma_r} e^{- \frac{r^2}{2 \sigma_r^2}} \tag{5}$$

Since the measurement noise is white, r (n) and $\bar{\underline{x}}$ (n) are independent; therefore,

$$f [\underline{x}(n), r(n)] = f_1 [\underline{x}(n)] \, f_2 [r(n)],$$

or,

$$f [\underline{x}(n), r(n)] = \frac{1}{4\pi^2 \sigma_r |M(n)|^{1/2}} e^{-1/2 \left\{ [\underline{x}(n) - \bar{\underline{x}}(n)] M^{-1}(n) [\underline{x}(n) - \bar{\underline{x}}(n)]^T + \frac{r^2}{\sigma_r^2} \right\}} \tag{6}$$

The estimated value of $\underline{x}$ (n) that maximizes the joint probability density function (the most probable value of $\underline{x}$ (n)) maximizes the argument of the exponential given in (6). This then is the Maximum Likelihood Estimator. The likelihood function is

$$L [\underline{x}(n)] = \frac{1}{2} [\underline{x}(n) - \bar{\underline{x}}(n)] M^{-1}(n) [\underline{x}(n) - \bar{\underline{x}}(n)]^T + \frac{1}{2} \frac{r^2}{\sigma_r^2} \tag{7}$$

Notice that minimizing (7) is the same as maximizing the argument of the exponential in (6).

From Fig. (1), the relationship between r (n), z (n), and $\underline{x}$ (n) can be derived by observation.

$$r(n) = z(n) - \underline{h}^T \underline{x}(n), \tag{8}$$

where $\underline{h}^T = [1, 0, 0]$. The substitution of (8) into (7) yields:

$$L[x(n)] = \frac{1}{2}[\underline{x}(n) - \overline{\underline{x}}(n)] M^{-1}(n) [\underline{x}(n) - \overline{\underline{x}}(n)]^T + \frac{1}{2\sigma_r^2}[z(n) - \underline{h}^T \underline{x}(n)]^2$$

The first variation of the likelihood function is

$$\delta L = d^T \underline{x}(n) M^{-1}(n) [\underline{x}(n) - \overline{\underline{x}}(n)] - d^T \underline{x}(n) \frac{1}{\sigma_r^2} \underline{h} [z(n) - \underline{h}^T \underline{x}(n)]. \tag{9}$$

The value of $\underline{x}$ (n) which causes L [$\underline{x}$ (n)] to be stationary can be derived from (9).

$$M^{-1}(n) \hat{\underline{x}}(n) - M^{-1}(n) \overline{\underline{x}}(n) - \frac{1}{\sigma_r^2} \underline{h}\, z(n) + \frac{1}{\sigma_r^2} \underline{h}\, \underline{h}^T \hat{\underline{x}}(n) = 0$$

$$\left[\frac{1}{\sigma_r^2} \underline{h}\, \underline{h}^T + M^{-1}(n)\right] \hat{\underline{x}}(n) = \left[\frac{1}{\sigma_r^2} \underline{h}\, \underline{h}^T + M^{-1}(n)\right] \overline{\underline{x}}(n)$$

$$+ \frac{1}{\sigma_r^2} \underline{h} [z(n) - \underline{h}^T \overline{\underline{x}}(n)]$$

Therefore,

$$\hat{\underline{x}} = \bar{\underline{x}}(n) + \psi(n) \frac{1}{\sigma_r^2} \underline{h} \; [z(n) - \underline{h}^T \bar{\underline{x}}(n)], \tag{10}$$

where

$$\psi^{-1}(n) = [\frac{1}{\sigma_r^2} \underline{h}\,\underline{h}^T + M^{-1}(n)]. \tag{11}$$

It will now be shown that $\psi(n)$ is the covariance of the estimate at the $n^{th}$ sample after the incorporation of the $n^{th}$ measurement according to (10). The time subscript will be dropped for this derivation. Let $\underline{e} = \hat{\underline{x}} - \underline{x}$, then

$$\underline{e} = \hat{\underline{x}} - \bar{\underline{x}} + \bar{\underline{x}} - \underline{x} \tag{12}$$

The substitution of (8) and (10) into (12) yields:

$$\underline{e} = \psi \frac{1}{\sigma_r^2} \underline{h} (\underline{h}^T \underline{x} + r - \underline{h}^T \bar{\underline{x}}) + \bar{\underline{x}} - \underline{x},$$

or,

$$\underline{e} = \frac{1}{\sigma_r^2} \psi \underline{h} r + (I - \frac{1}{\sigma_r^2} \psi \underline{h}\,\underline{h}^T)(\bar{\underline{x}} - \underline{x}) \tag{13}$$

From (13) P can be expressed as

$$P = \overline{\underline{e}\,\underline{e}^T} = \frac{1}{\sigma_r^2} \psi \underline{h}\,\underline{h}^T \psi^T + (I - \frac{1}{\sigma_r^2} \psi \underline{h}\,\underline{h}^T) M (I - \frac{1}{\sigma_r^2} \psi \underline{h}\,\underline{h}^T)^T \tag{14}$$

Premultiplying (11) by $\psi$ and then postmultiplying by M yields .

$$M = \psi + \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T M,$$

or

$$(I - \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T) M = \psi \qquad (15)$$

The substitution of (15) into (14) yields:

$$P = \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T \psi^T + \psi (I - \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T)^T,$$

or

$$P = \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T \psi^T - \frac{1}{\sigma_r^2} \psi \underline{h} \underline{h}^T \psi^T + \psi.$$

Therefore,

$$P = \psi .$$

## Summary

The correction scheme is summarized below.

(1) Precompute and store in the flight computer the filter weights for the maximum likelihood estimator as follows.

(a) Let $\underline{w}(n) = \frac{1}{\sigma_r^2} P(n) \underline{h}$, where the initial covariance $P(0)$ is given.

(b) Extrapolate the covariance matrix using (3)

$$M(n+1) = \phi P(n) \phi^T + \Gamma Q(n) \Gamma^T \quad (3)$$

(c) Calculate the covariance matrix, after the incorporation of the measurement using (11)

$$P(n+1) = [\frac{1}{\sigma_r^2} \underline{h}\, \underline{h}^T + M^{-1}(n+1)] \quad (11)$$

(d) Calculate the n + 1 weighting vector from

$$\underline{w}(n+1) = \frac{1}{\sigma_r^2} P(n+1) \underline{h}$$

Repeat (b) through (d) until a sufficient number of filter weights have been calculated.

(2) In the flight computer

(a) Extrapolate the state using (2)

$$\underline{\bar{x}}(n+1) = \phi \underline{\hat{x}}(n) + \Gamma \underline{\bar{u}}(n) \quad (2)$$

where $\underline{\hat{x}}(0)$ is given.

(b) Incorporate the measurement using (10).

$$\underline{\hat{x}}(n+1) = \underline{\bar{x}}(n+1) + \underline{w}(n+1) \; [z(n+1) - \underline{h}^T \underline{\bar{x}}(n+1)] \quad (10)$$

(e) Apply the thrust vector misalignment signal $\delta_{cor}(n+1) = \hat{x}_3(n+1) = \hat{\delta}_b(n+1)$

A block diagram of this correction scheme is shown below.

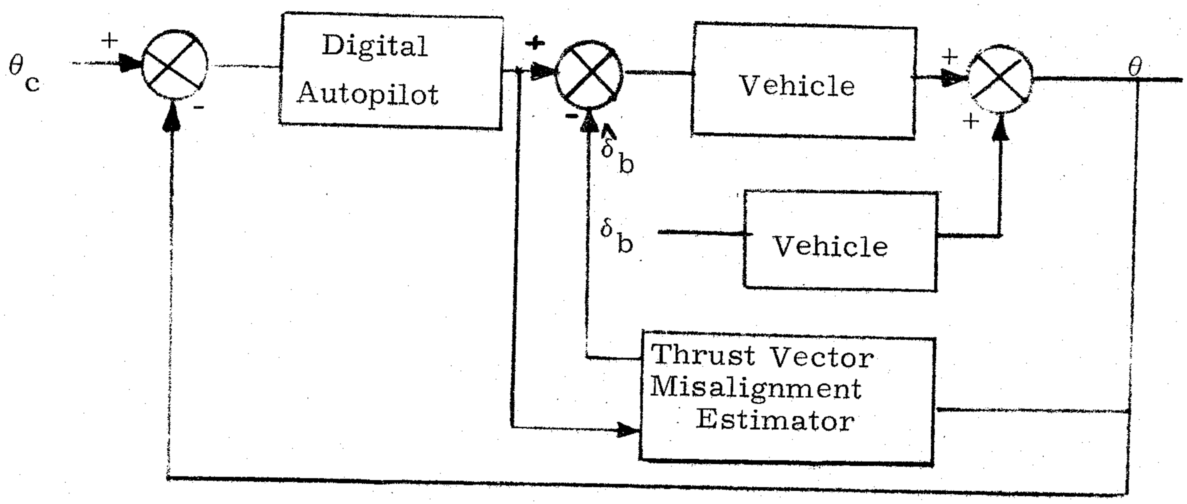



Figure 2. A Block Diagram of the Implementation for the Recursive Correction Scheme

Equations (2) and (10) can be simplified for programming as follows: Equation (2) written out is

$$\bar{\theta}(n+1) = \hat{\theta}(n) + \Delta T\, \dot{\hat{\theta}}(n) + k\frac{\Delta T^2}{2}\hat{\delta}_b(n) + k\frac{\Delta T^2}{2}[\delta_c(n) - \hat{\delta}_b(n)] \tag{2a}$$

$$\bar{\dot{\theta}}(n+1) = \dot{\hat{\theta}}(n) + k\,\Delta T\,\hat{\delta}_b(n) + k\,\Delta T\,[\delta_c(n) - \hat{\delta}_b(n)] \tag{2b}$$

$$\bar{\delta}_b(n+1) = \hat{\delta}_b(n) \tag{2c}$$

, or,

$$\left.\begin{aligned} \bar{\theta}(n+1) &= \hat{\theta}(n) + \Delta T\,\dot{\hat{\theta}}(n) + k\frac{\Delta T^2}{2}\delta_c(n) \\ \bar{\dot{\theta}}(n+1) &= \dot{\hat{\theta}}(n) + k\,\Delta T\,\delta_c(n) \\ \bar{\delta}_b(n+1) &= \hat{\delta}_b(n) \end{aligned}\right\} \quad (2')$$

Substituting (2') into (10) yields:

$$\hat{\theta}(n+1) = \bar{\theta}(n+1) + w_0\, c\,(n+1) \tag{16}$$

$$\dot{\hat{\theta}}(n+1) = \dot{\bar{\theta}}(n+1) + w_1\, c\,(n+1) \tag{17}$$

$$\delta_b(n+1) = \delta_b(n) + w_2\, c\,(n+1), \tag{18}$$

where $w(n+1) = [w_0(n+1),\ w_1(n+1),\ w_2(n+1)]^T$

and $c(n+1) = z(n+1)\ \underline{h}^T\ \underline{x}(n+1) = z(n+1) - \bar{\theta}(n+1)$.

## Results

This correction scheme has been applied to the CSM and the CSM - LM vehicles. A total thrust vector misalignment of $1^{\circ}$ was used for each simulation.

Plots of the attitude, engine angle, and the velocity error verses time for the CSM vehicle are shown in Figs. (3) thru (5). Corresponding plots for the CSM - LM vehicle are shown in Figs. (6) thru (9).

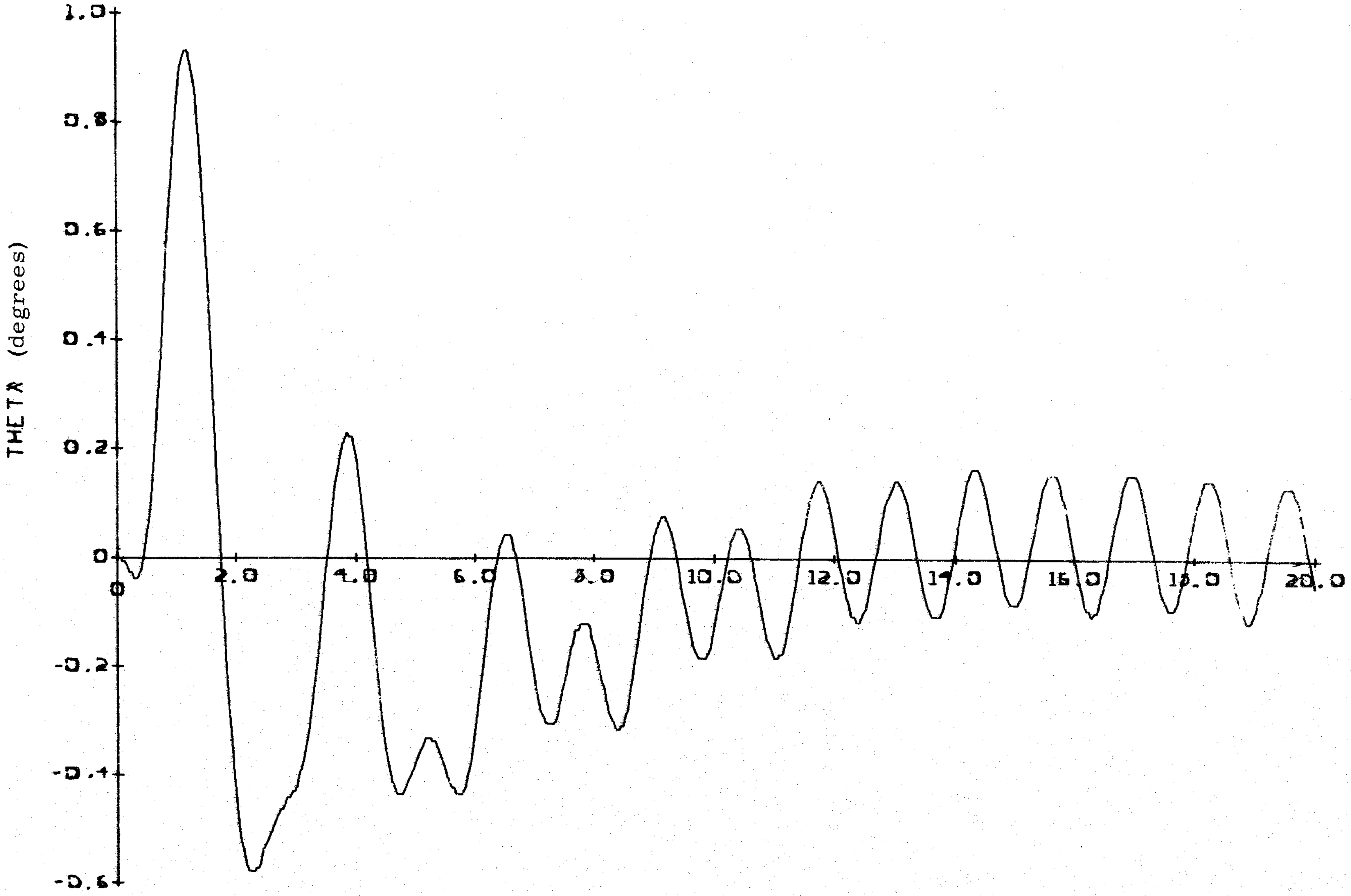



Fig. 3. A plot of attitude error versus times for the CSM.

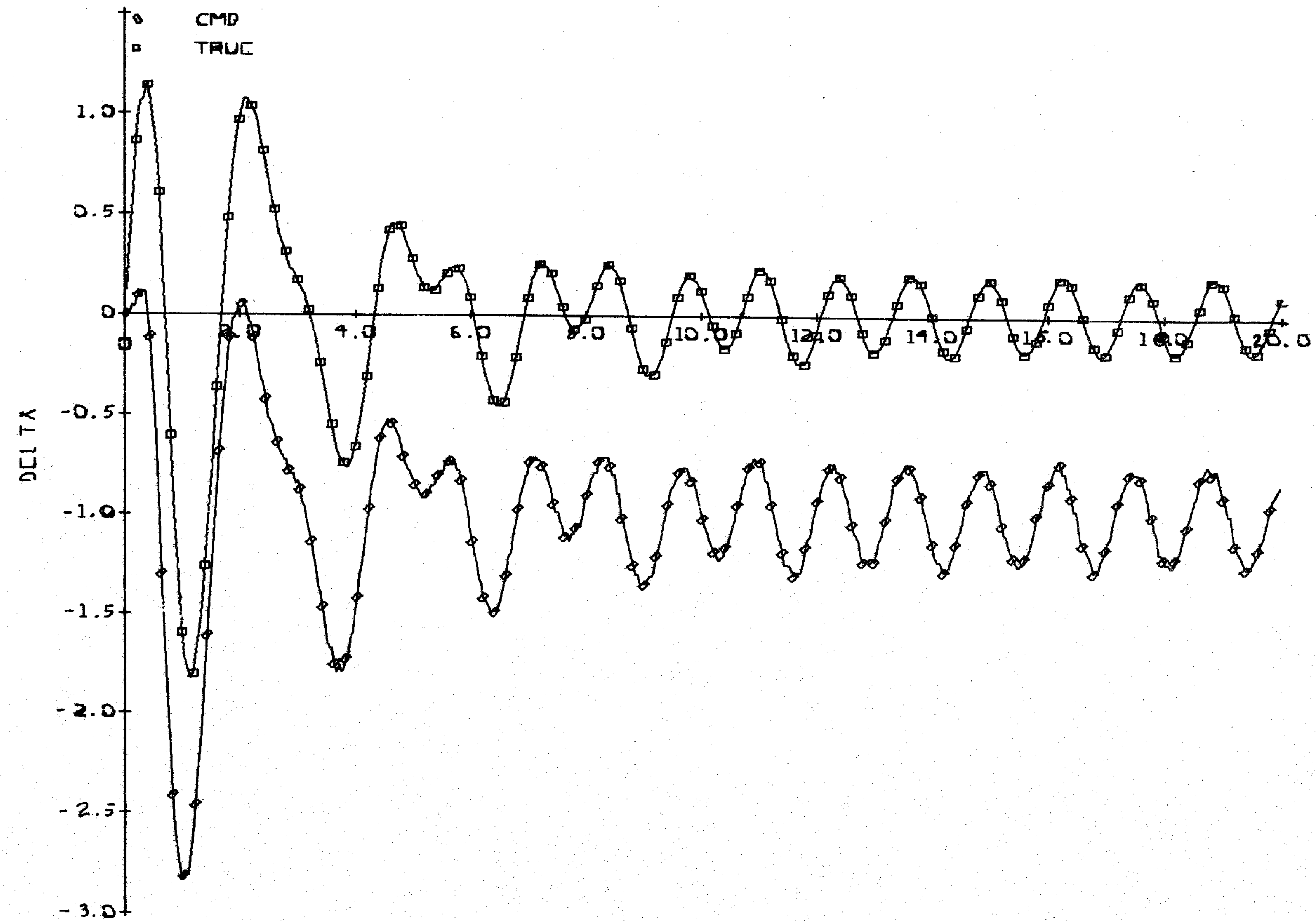



Fig. 4. A plot of the commanded and the actual engine position versus time for the CSM.

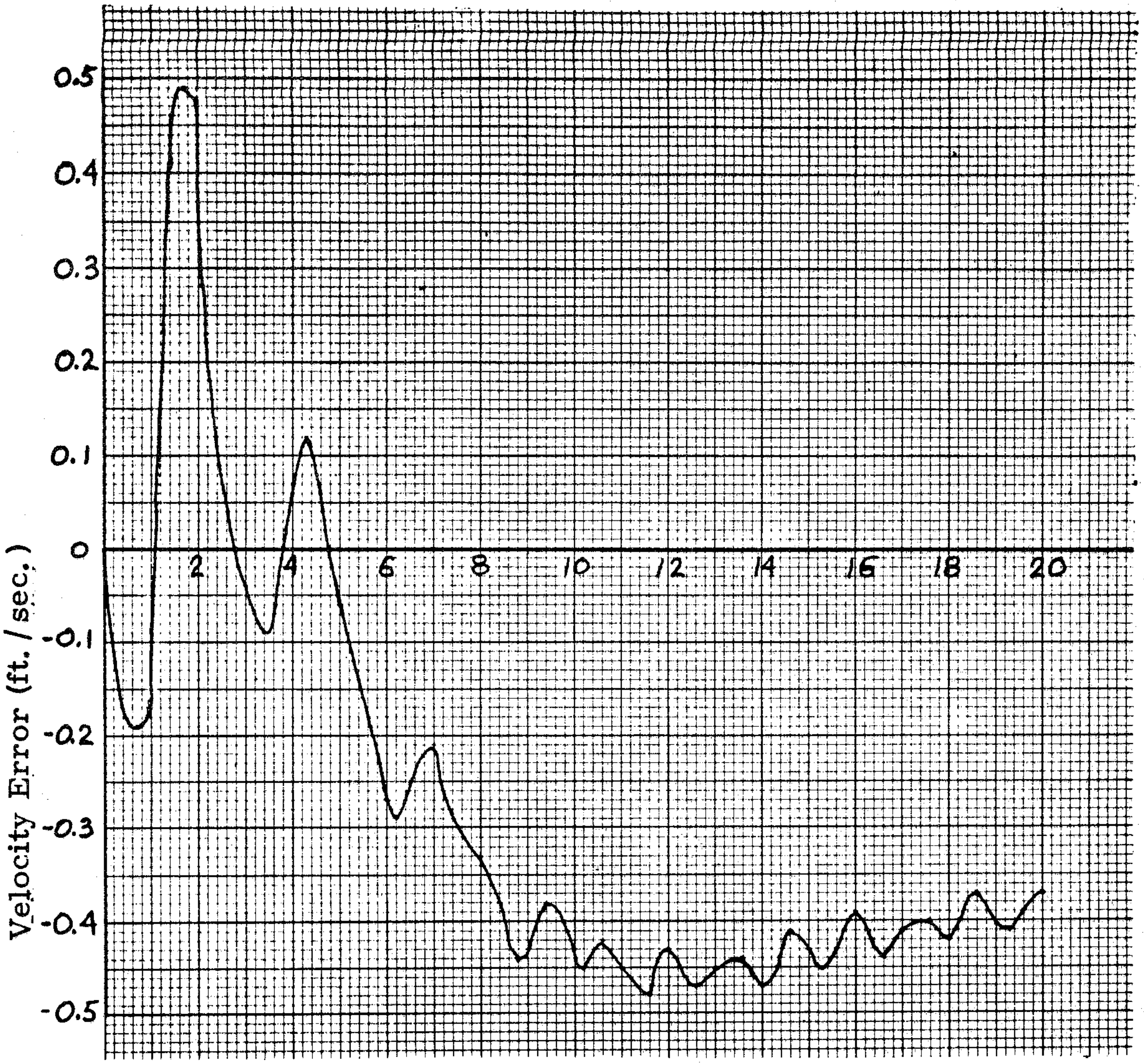



Fig. 5. A plot of the velocity error versus time for the CSM.

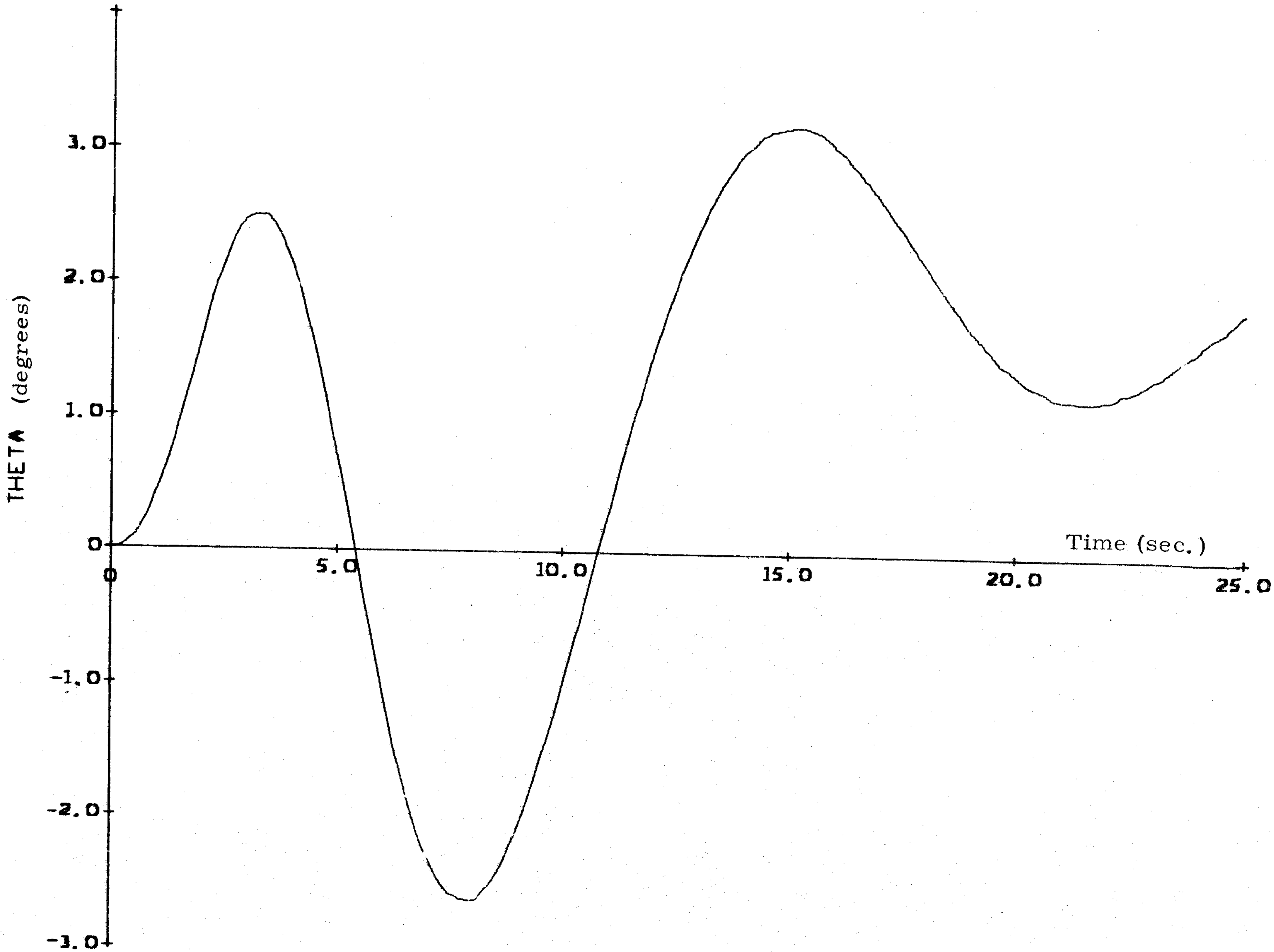





Fig. 6. A plot of attitude error versus time for the CSM-LM.

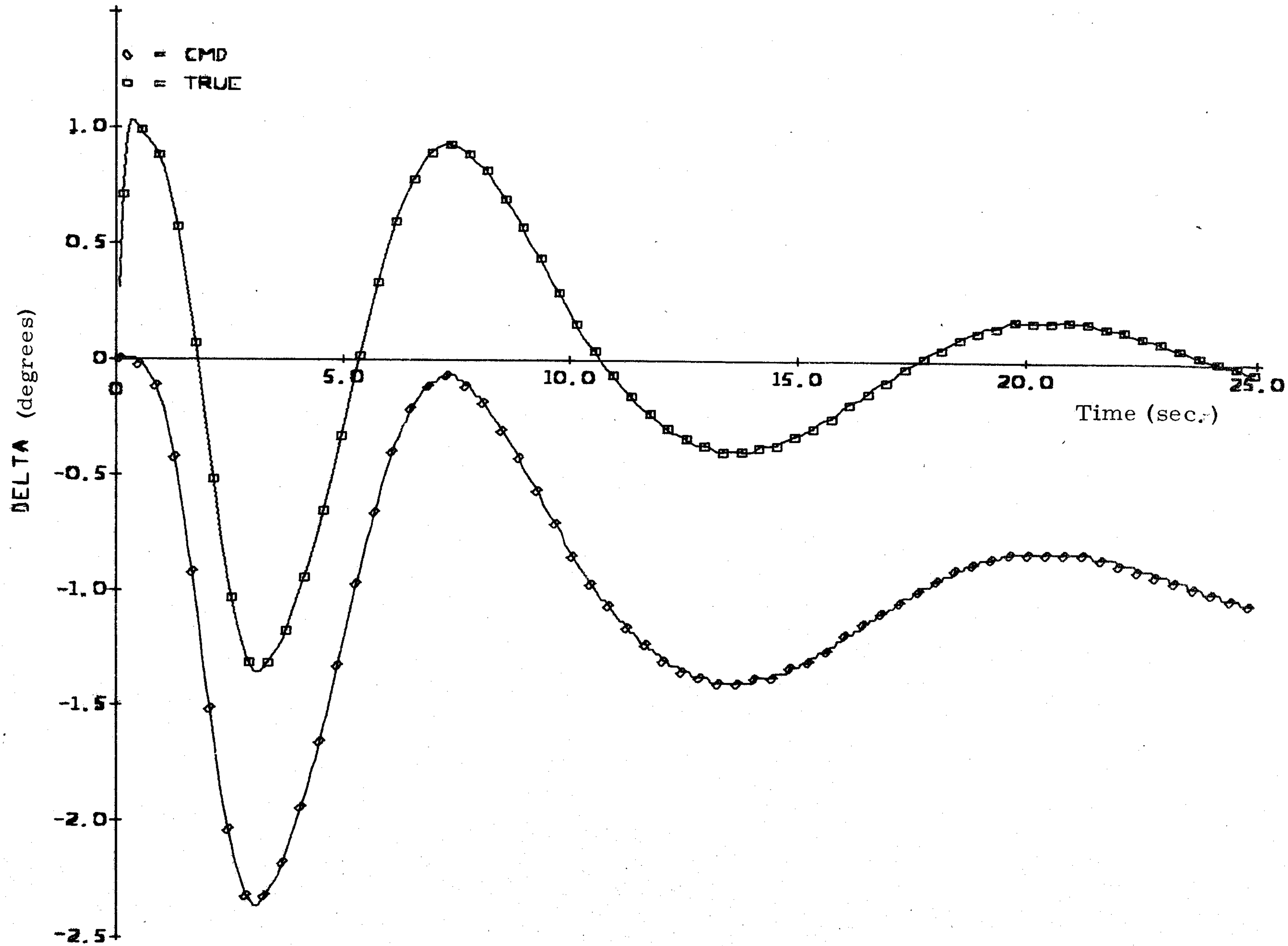



Fig. 7. A plot of the commanded and the actual engine position versus time for the CSM-LM.

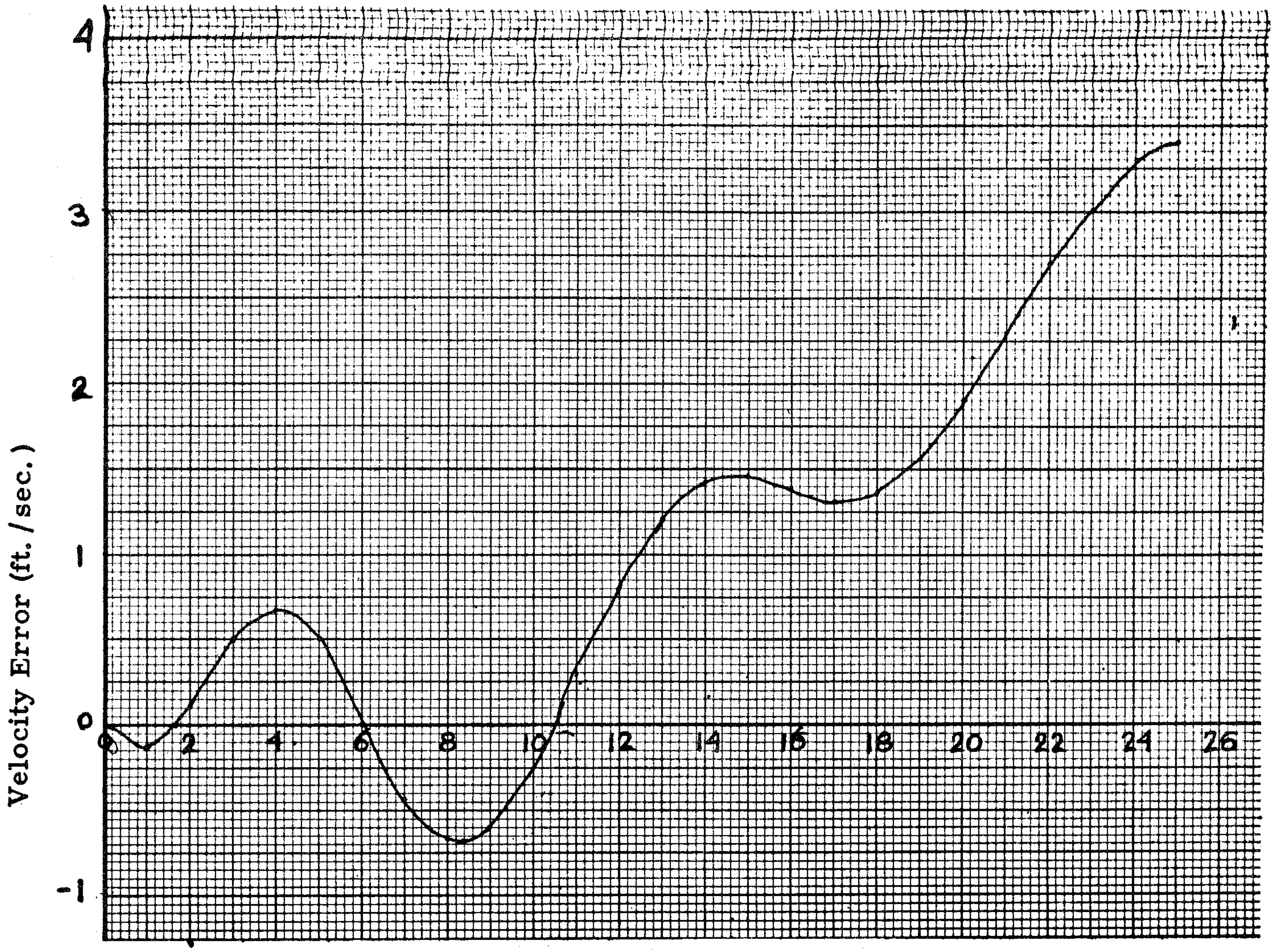



Fig. 8. A plot of the velocity error versus time for the CSM-LM.

## Conclusions

It was found that the attitude errors were less than 3 degrees for the cases shown in Figs. (3) and (6). The curves shown in Figs. (5) and (8) indicate that the velocity error for a 25 second burn without steering never exceeds 0.5 ft/sec. in the case of the CSM vehicle, and 4 ft/sec. for the CSM - LM vehicle.

Figures (3) and (6) indicate that there might be a relative stability problem produced by the addition of the corrective signal. However, the resulting system does appear to be stable in an absolute sense . The curves in Figs. (4) and (7) show that the correction signal rapidly approaches the value required to cancel the the initial thrust misalignment and then oscillates about this value. Therefore, the correction process could be terminated early by supplying a constant bias equal to the average value of the oscillating signal after approximately 10 seconds. This would eliminate the stability problem.

The interaction between the estimator and the original system is being investigated to determine the resulting stability margins. It is hoped that these studies will lead to approaches for improving the overall system stability through revised estimation procedures, and better coordination between the autopilot design and the estimator selection.

# References

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