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Space Guidance Analysis Memo #9

To: SGA Distribution
From: R. H. Battin
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Subject: Precise Circumlunar Trajectory Calculations

The purpose of this memorandum is to present an alternate and potentially more reliable method of obtaining a precision circumlunar trajectory than the procedure described in Section 4 of report R-353. As before, the pieced conics solution provides the first approximation to be used in an iteration scheme, but the new method described here should be free of any troublesome convergence problems which frequently caused difficulties in the original approach.

The following quantities are obtained from the pieced conics approximation and are invariant during the course of the iteration:

(1) $\underline{r}_L$, the launch position vector;
(2) t_L, the time of launch;
(3) t_A, the time of arrival at the lunar sphere of influence;
(4) $t_D = t_A + t_S$, the time of departure from the lunar sphere of influence;
(5) $\underline{r}_R$, the return vacuum perigee position vector;
(6) t_R, the time of return to the vacuum perigee position.

An outline of the calculational procedure follows:

(1) Let $\underline{r}_{TA}$ and $\underline{r}_{TD}$ be vectors from the center of the Earth to the points on the lunar sphere of influence where the trajectory arrives and departs, respectively. During the course

of the iteration these vectors will be altered but initially they are taken from the pieced conics approximation.

(2) With $\underline{r}_L$ and $\underline{r}_{TA}$ fixed, a precise trajectory connecting these points in a given time $t_{FL} = t_A - t_L$ is readily obtained using a procedure suggested by Th. Godal. First a Keplerian orbit is determined by the method described in SGA Memo No. 3. Then the effects of perturbations are computed by integrating the equations of motion using as initial conditions $\underline{r}_L$ and the velocity of the Keplerian orbit at $\underline{r}_L$. Thus, a position deviation from $\underline{r}_{TA}$ at time t_A is determined and used to shift the "aiming point" of a second Keplerian reference orbit. Again the effects of perturbations are computed and the process repeated until satisfactory convergence is obtained.

(3) Step (2) is repeated using $\underline{r}_R$, $\underline{r}_{TD}$ and $t_{FR} = t_R - t_D$.

(4) Step (2) is repeated using $\underline{r}_{TML}$ and $\underline{r}_{TMR}$, the position vectors relative to the Moon at the sphere of influence, and the time t_S spent within the sphere.

(5) As a result of Steps (2)-(4) a piece-wise continuous precise circumlunar trajectory is obtained having velocity discontinuities at the sphere of influence. In order to eliminate these two velocity mismatches, the following perturbation matrices are computed as solutions of differential equations

(a) $C_L^{-1}(t_A)$ obtained from

$$\frac{d}{dt} C_L^{-1} + C_L^{-1} G C_L^{-1} = I$$

$$C_L^{-1}(t_L) = O$$

(b) $C_R^{*-1}(t_D)$ obtained from

$$\frac{d}{dt} C_R^{*-1} + C_R^{*-1} G C_R^{*-1} = I$$

$$C_R^{*-1}(t_R) = O$$

(c) $C_h^{-1}(t_D)$ obtained from

$$\frac{d}{dt} C_h^{-1} + C_h^{-1} G C_h^{-1} = I$$

$$C_h^{-1}(t_A) = O$$

(d) $C_h^{*-1}(t_A)$ obtained from

$$\frac{d}{dt} C_h^{*-1} + C_h^{*-1} G C_h^{*-1} = I$$

$$C_h^{*-1}(t_D) = O$$

(e) $R_h(t_D)$ obtained from

$$\frac{d^2}{dt^2} R_h = G R_h$$

$$R_h(t_A) = O$$

$$\frac{dR_h}{dt}(t_A) = I$$

(f) $R_h^*(t_A)$ obtained from

$$\frac{d^2}{dt^2} R_h^* = G R_h^*$$

$$R_h^*(t_D) = O$$

$$\frac{dR_h}{dt}(t_D) = I$$

In these calculations the matrices O and I are the three dimensional zero and identity matrices. The elements of the matrix G are the partial derivatives of the components of gravity with respect to the components of position.

(6) Suppose $\underline{r}_{TA}$ and $\underline{r}_{TD}$ are shifted by the amounts $\delta \underline{r}_A$ and $\delta \underline{r}_D$ respectively. The velocity vectors at and exterior to the sphere must change by

$$\delta \underline{v}_L (t_A) = C_L (t_A) \delta \underline{r}_A$$

$$\delta \underline{v}_R (t_D) = C_R^* (t_D) \delta \underline{r}_D$$

if $\underline{r}_L$, $\underline{r}_R$ and the times of flight are to remain invariant. In like manner the velocity vectors at and interior to the sphere will change by

$$\delta \underline{v}_h (t_A) = R_h^{-1} (t_D) \delta \underline{r}_D + C_h^* (t_A) \delta \underline{r}_A$$

$$\delta \underline{v}_h (t_D) = C_h (t_D) \delta \underline{r}_D + R_h^{*-1} (t_A) \delta \underline{r}_A$$

if the time of flight is invariant. The problem is, of course, to determine $\delta \underline{r}_D$ and $\delta \underline{r}_A$ so that the resulting velocity changes will exactly cancel the original velocity differences at the points of discontinuity.

Initially, the velocity mismatch at $\underline{r}_{TA}$ is

$$\Delta \underline{v}_A = \underline{v}_{TML} - \underline{v}_h (t_A)$$

and at $\underline{r}_{TD}$ the mismatch is

$$\Delta \underline{v}_D = \underline{v}_h (t_D) - \underline{v}_{TMR}$$

Thus, $\delta \underline{r}_A$ and $\delta \underline{r}_D$ must be chosen such that

$$\delta \underline{v}_L (t_A) - \delta \underline{v}_h (t_A) + \Delta \underline{v}_A = O$$

$$\delta \underline{v}_h (t_D) - \delta \underline{v}_R (t_D) + \Delta \underline{v}_D = O$$

The resulting solution may be written as

$$\delta \underline{r}_A = R_h^* (t_A) \left[R_h (t_D) A^{-1} (t_A) R_h^* (t_A) - B (t_D) \right]^{-1} R_h (t_D) \Delta \underline{v}_A$$

$$+ A (t_A) \left[A (t_A) - R_h^* (t_A) B^{-1} (t_D) R_h (t_D) \right]^{-1} R_h^* (t_A) \Delta \underline{v}_D$$

$$\delta \underline{r}_D = B (t_D) \left[B (t_D) - R_h (t_D) A^{-1} (t_A) R_h^* (t_A) \right]^{-1} R_h (t_D) \Delta \underline{v}_A$$

$$+ R_h (t_D) \left[R_h^* (t_A) B^{-1} (t_D) R_h (t_D) - A (t_A) \right]^{-1} R_h^* (t_A) \Delta \underline{v}_D$$

where, for convenience, the matrices $A (t_A)$ and $B (t_D)$ are defined as

$$A (t_A) = C_h^{*-1} (t_A) \left[C_h^{*-1} (t_A) - C_L^{-1} (t_A) \right]^{-1} C_L^{-1} (t_A)$$

$$B (t_D) = C_R^{*-1} (t_D) \left[C_R^{*-1} (t_D) - C_h^{-1} (t_D) \right]^{-1} C_h^{-1} (t_D)$$

(7) Repeat from the beginning with $\underline{r}_{TA}$ replaced by $\underline{r}_{TA} + \delta \underline{r}_A$ and $\underline{r}_{TD}$ replaced by $\underline{r}_{TD} + \delta \underline{r}_D$ until satisfactory velocity continuity is obtained.