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Space Guidance Analysis Memo # 8-67

TO: SGA Distribution

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DATE: May 1, 1967

SUBJECT: Explicit Universal Series Solutions for the Universal Variable x.

REFERENCE: A Unified Method of Generating Conic Sections, William Marscher, MIT/IL R-479, February, 1965.

I. Introduction

In the reference, a set of seven equations are given which are universal (applicable without change to the ellipse, the parabola, and the hyperbola), and by means of which the following conic problems may be solved: Kepler's Problem, Lambert's Problem, the Reentry Problem, the Time-Theta Problem, and the Time-Radius Problem. The universal solution of all of these except Kepler's Problem requires the iterative determination of the universal variable x from the equation:

$$\cot \frac{\theta}{2} = \frac{r_0 \left[1 - \alpha x^2 S(\alpha x^2)\right]}{\sqrt{p}\, x\, C(\alpha x^2)} + \cot \gamma_0 ,$$

given the values of the other parameters.*

This paper presents a set of explicit series solutions for x, given the same parameters as in the above equation, namely the semi-latus rectum and the reciprocal of the semi-major axis of the conic, the initial position distance and the initial flight path angle, and the transfer angle.

* A set of three non-iterative, but non-universal, solutions for x may be obtained, one for each type of conic section, but the elliptic and hyperbolic expressions become undefined as parabolic eccentricity is approached, causing computational problems, although x itself is theoretically well defined.

Each series of the set is universal and converges, and the rate of convergence may be easily examined analytically.

II. Notation (The notation agrees with that in the reference).

r_0 = initial position vector magnitude

γ_0 = initial flight path angle (measured from local vertical)

p = semi-latus rectum of conic

α = reciprocal of semi-major axis of conic (negative for hyperbolas)

θ = transfer angle = final true anomaly minus initial true anomaly

ΔE = final eccentric anomaly minus initial eccentric anomaly (ellipse)

ΔG = hyperbolic equivalent to ΔE

x = universal variable = $\frac{\Delta E}{\sqrt{\alpha}}$ for ellipse, $\frac{\Delta G}{\sqrt{-\alpha}}$ for hyperbola

III. Explicit Series for x

$$\text{Let } W_1 = \frac{\cot\frac{\theta}{2} - \cot\gamma_0}{r_0/\sqrt{p}}, \quad (0 < \theta < 2\pi).$$

$$\text{Let } W_n = +\sqrt{W_{n-1}^2 + \alpha} + W_{n-1}, \quad (n \geq 2).$$

Then, for each $n \geq 3$,

$$x = \frac{2^n}{W_n}\left[1 - \frac{1}{3}\frac{\alpha}{W_n^2} + \frac{1}{5}\frac{\alpha^2}{W_n^4} - \frac{1}{7}\frac{\alpha^3}{W_n^6} + \frac{1}{9}\frac{\alpha^4}{W_n^8} - \dots\right]$$

The series for each $n \geq 3$ is universal and converges to x for $0 < \theta < 2\pi$.

In the cases where θ is very nearly 0^o or 360^o, there may be numerical difficulties in calculating W_n and using it in the series, and one should work directly in terms of its reciprocal, determining it by the following method: First the reciprocal of W_1 is computed by

$$\frac{1}{W_1} = \frac{(\sin\theta)\, r_0/\sqrt{p}}{1 + \cos\theta - \sin\theta \cot\gamma_0}$$

Next, the intermediate quantities V_n should be calculated using the recursion relation:

$$V_n = \sqrt{V_{n-1}^2 + \alpha \left(\frac{1}{W_1}\right)^2} + V_{n-1} \qquad (n \geq 2)$$

where $V_1 = 1$. Then the reciprocal of W_n is given by

$$\frac{1}{W_n} = \left(\frac{1}{W_1}\right)/V_n.$$

IV. Derivation

The starting point for the derivation of the series solutions for x are Eqs. (18a) and (19a) of the reference, namely:

$$\cot\frac{\theta}{2} = \frac{r_0}{\sqrt{p}} \sqrt{\alpha} \cot\frac{\Delta E}{2} + \cot\gamma_0 \qquad \text{(elliptic)}$$

$$\cot\frac{\theta}{2} = \frac{r_0}{\sqrt{p}} \sqrt{-\alpha} \coth\frac{\Delta G}{2} + \cot\gamma_0 \qquad \text{(hyperbolic)}$$

Let

$$W_1 = \frac{\cot \frac{\theta}{2} - \cot \gamma_0}{r_0/\sqrt{p}} .$$

Then we have

$$\cot \frac{\Delta E}{2} = \frac{W_1}{\sqrt{\alpha}} \text{ (elliptic)}; \quad \coth \frac{\Delta G}{2} = \frac{W_1}{\sqrt{-\alpha}} \text{ (hyperbolic)}$$

The half-angle formulas for the circular and hyperbolic contangent are:

$$\cot \frac{\phi}{2} = \frac{1 + \cos \phi}{\sin \phi} = \csc \phi + \cot \phi = \pm \sqrt{\cot^2 \phi + 1} + \cot \phi$$

where the positive sign is taken if ϕ is in the first or second quadrants, and the negative sign if ϕ is in the third or fourth quadrants, and

$$\coth \frac{\phi}{2} = \frac{1 + \cosh \phi}{\sinh \phi} = \operatorname{csch} \phi + \coth \phi = \pm \sqrt{\coth^2 \phi - 1} + \coth \phi$$

where the positive sign is taken if ϕ is positive, and the negative sign if ϕ is negative.

We assume that $0 \leq \theta \leq 2\pi$, which implies that $0 \leq \Delta E \leq 2\pi$ and that $0 \leq \Delta G$.

Substituting $\cot \frac{\Delta E}{2}$ and $\coth \frac{\Delta G}{2}$ into the half-angle formulas, we obtain

$$\cot \frac{\Delta E}{4} = + \sqrt{\frac{W_1^2}{\alpha} + 1} + \frac{W_1}{\sqrt{\alpha}}$$

$$\coth \frac{\Delta G}{4} = + \sqrt{\frac{W_1^2}{-\alpha} - 1} + \frac{W_1}{\sqrt{-\alpha}}$$

If we let $W_n = \sqrt{W_{n-1}^2 + \alpha} + W_{n-1}$, then

$$\cot \frac{\Delta E}{4} = \frac{W_2}{\sqrt{\alpha}}, \quad \coth \frac{\Delta G}{4} = \frac{W_2}{\sqrt{-\alpha}} ,$$

and by substituting into the half-angle formulas to get the 8th, 16th, 32nd, etc., angles, we obtain

$$\cot \frac{\Delta E}{2^n} = \frac{W_n}{\sqrt{\alpha}}, \quad \coth \frac{\Delta G}{2^n} = \frac{W_n}{\sqrt{-\alpha}} .$$

It is to be noted that the positive sign on the radical in the half-angle formulas is taken every time since $\frac{\Delta E}{2^n}$ is in the first or second quadrant for $n \geq 1$, and $\frac{\Delta G}{2^n}$ is always positive.

Now $\Delta E = x \sqrt{\alpha}$ and $\Delta G = x \sqrt{-\alpha}$. Thus,

$$x \sqrt{\alpha} = 2^n \operatorname{arccot} \frac{W_n}{\sqrt{\alpha}} \quad \text{(elliptic)}$$

$$x \sqrt{-\alpha} = 2^n \operatorname{arccoth} \frac{W_n}{\sqrt{-\alpha}} \quad \text{(hyperbolic)}$$

The inverse circular and hyperbolic cotangents have the expansions

$$\operatorname{arccot} z = \frac{1}{z} - \frac{1}{3z^3} + \frac{1}{5z^5} - \frac{1}{7z^7} + \ldots..$$

which converges for $z \geq 1$, and

$$\operatorname{arccoth} z = \frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \frac{1}{7z^7} + \ldots..$$

which converges for $z > 1$.

Thus, using these expansions,

$$x = \frac{2^n}{W_n}\left[1 - \frac{1}{3}\left(\frac{\sqrt{\alpha}}{W_n}\right)^2 + \frac{1}{5}\left(\frac{\sqrt{\alpha}}{W_n}\right)^4 - \frac{1}{7}\left(\frac{\sqrt{\alpha}}{W_n}\right)^6 + \cdots\cdots\right] \quad \text{(elliptic)}$$

$$x = \frac{2^n}{W_n}\left[1 + \frac{1}{3}\left(\frac{\sqrt{-\alpha}}{W_n}\right)^2 + \frac{1}{5}\left(\frac{\sqrt{-\alpha}}{W_n}\right)^4 + \frac{1}{7}\left(\frac{\sqrt{-\alpha}}{W_n}\right)^6 + \cdots\cdots\right] \quad \text{(hyperbolic)}$$

which reduce to the single series:

$$x = \frac{2^n}{W_n}\left[1 - \frac{1}{3}\frac{\alpha}{W_n^2} + \frac{1}{5}\frac{\alpha^2}{W_n^4} - \frac{1}{7}\frac{\alpha^3}{W_n^6} + \cdots\cdots\right]$$

as was to be shown.

V. Convergence Questions

From the successive half-angle substitutions, we have already shown in the last section that for $n \geq 1$,

$$\cot\frac{\Delta E}{2^n} = \frac{W_n}{\sqrt{\alpha}} \quad \text{and} \quad \coth\frac{\Delta G}{2^n} = \frac{W_n}{\sqrt{-\alpha}}$$

Thus the argument $\left(\frac{\alpha}{W_n^2}\right)$ in the power series is given by:

$$\frac{\alpha}{W_n^2} = \begin{cases} \tan^2\frac{\Delta E}{2^n} & \text{(elliptic case)} \\ -\tanh^2\frac{\Delta G}{2^n} & \text{(hyperbolic case)} \end{cases}$$

Consequently, in the elliptic case, since $0 \leq \Delta E < 2\pi$, we prove by the ratio test that the series converge provided that

$$\tan^2 \frac{\Delta E}{2^n} < 1 \text{ or } \frac{\Delta E}{2^n} < \frac{\pi}{4} \text{ or } n \geq 3.$$

If ΔE is restricted to be less than π (or to be less than $\pi/2$), then the series also converges for $n = 2$ (or $n = 1$, respectively). However, $n = 3$ is the first series for x which converges for all ΔE between 0 and 2π.

In the hyperbolic case, we prove again by the ratio test that the series always converges for $n \geq 1$ for all ΔG, since the hyperbolic tangent is always less than one.

A good idea of the rate of convergence of the series is also given by the expression for the argument (α / W_n^2) of the power series: for all p, α, r_0, γ_0, and for all θ between 0 and 2π, we have

$$\left| \frac{\alpha}{W_n^2} \right| < \tan^2 \frac{2\pi}{2^n} \qquad \text{(elliptic)}$$

$$\left| \frac{\alpha}{W_n^2} \right| < \tanh^2 \frac{\Delta G_{max}}{2^n} \qquad \text{(hyperbolic)}$$

where ΔG_{max} (an upper bound on ΔG) is determined from other considerations. From these inequalities, one can see directly that a series with a larger value of n converges more rapidly than one with a smaller value of n.

In the parabolic case, $\alpha = 0$, and the series converges in one term, provided $W_n \neq 0$. To show that this is true, we begin by showing $W_1 \neq 0$. Now

$$W_1 = \frac{\cot(\theta/2) - \cot \gamma_0}{r_0/\sqrt{p}} \text{ from the definition.}$$

But p equals twice the pericenter radius and is thus non-zero. Now, for a parabola, the flight path angle at any point and the true anomaly of the point are related by the equation $\cot \gamma = \tan f/2$. Thus

$$\cot \frac{\theta}{2} - \cot \gamma_0 = \cot \frac{\theta}{2} - \tan \frac{f_0}{2} =$$

$$= \cot \left(\frac{f_1 - f_0}{2}\right) - \tan \frac{f_0}{2} = \frac{\cos f_1/2}{(\sin \frac{\theta}{2})(\cos \frac{f_0}{2})}$$

using standard trigonometric identities. But

$$\frac{\cos f_1/2}{(\sin \frac{\theta}{2})(\cos \frac{f_0}{2})}$$

is zero if and only if f_1 equals π or $-\pi$, which cannot occur. Thus $W_1 \neq 0$ for a parabola. Since $\alpha = 0$, $W_n = 2^{n-1} W_1$, and consequently $W_n \neq 0$ $(n \geq 1)$ for a parabola. Thus, for a parabola, the series reduces to its first term:

$$x = \frac{2^n}{W_n} = \frac{2}{W_1}$$

which agrees with the value obtained when x is calculated directly (cf. Eq. (20a) in the reference).