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Space Guidance Analysis Memo # 7-67

TO:	Distribution
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SUBJECT:	Correcting Fixed Dimension Kalman Gain Vectors to Account for Colored Measurement Noise

Introduction

The usual method of performing estimation when the measurements contain colored (non white) noise is to estimate the correlated part of the measurement noise by augmenting it to the state. The purpose of this memo is to present a method for changing the Kalman gains to account for the colored noise without increasing the dimension of the state. While the method presented here cannot provide as good an estimate as the augmented state technique, it should give better results than not using the colored noise information at all.

The motivation for this analysis is a problem which has arisen when performing state estimation in the LM digital autopilot. Slosh, which is not included in the on board dynamical model of the vehicle, is appearing in the measurements as additive correlated noise. To avoid the additional computation required by the augmented state technique, a method is needed to determine how to change the preprogrammed Kalman gain vectors which does not increase the dimension of the state, but which does recognize that the measurements contain correlated noise. Such a method is presented here.

Derivation

The object here is to find the weighting vector, $\underline{w}_k$, which minimizes the mean squared error in the estimate:

$$\hat{\underline{x}}_k = \hat{\underline{x}}'_k + \underline{w}_k \left(z_k - \underline{h}_k^T \hat{\underline{x}}'_k\right) \tag{1}$$

when the scalar measurement is corrupted by both white and correlated noise:

$$z_k = \underline{h}_k^T \underline{x}_k + c_k + n_k$$

where

$$\begin{aligned} \overline{c_k c_i} &= C_{ki} \\ \overline{n_k n_i} &= R_k \qquad k = i \\ \overline{n_k n_i} &= 0 \qquad k \neq i \\ \overline{c_k n_i} &= 0 \end{aligned} \tag{2}$$

From Eq. (1) and the definition of error:

$$\underline{e}_k \triangleq \hat{\underline{x}}_k - \underline{x}_k$$

we may write:

$$\underline{e}_k = \underline{e}_k' + \underline{w}_k \left(c_k + n_k - \underline{h}_k^T \underline{e}_k'\right)$$

$$= \left(I - \underline{w}_k \underline{h}_k^T\right) \underline{e}'_k + \underline{w}_k \left(c_k + n_k\right) \tag{3}$$

The mean squared error is equivalent to the trace of the corresponding covariance matrix:

$$J = \overline{\underline{e}_k^T \underline{e}_k} = tr(P_k) = tr(\overline{\underline{e}_k \underline{e}_k^T}) \tag{4}$$

Using Eqs. (2) and (3) plus the fact that the current white noise component of the measurement is uncorrelated with all values of the error which are based only on past measurements we may write:

$$\begin{aligned} P_k = & (I - \underline{w}_k \underline{h}_k^T) P_k' (I - \underline{w}_k \underline{h}_k^T)^T + \underline{w}_k (C_{kk} + R_k) \underline{w}_k^T \\ & + (I - \underline{w}_k \underline{h}_k^T) \overline{\underline{e}_k' c_k} \underline{w}_k^T + \underline{w}_k \overline{c_k \underline{e}_k'^T} (I - \underline{w}_k \underline{h}_k^T)^T \end{aligned} \tag{5}$$

We will now evaluate the term $\underline{e}_k' c_k$. Since the estimator is linear we may write:

$$\underline{e}_k' = \Psi(k, 0) \underline{e}_0 + \sum_{i=i}^{k-1} \Psi(k, i+1) \Phi(i+1, i) \underline{w}_i (c_i + n_i) \tag{6}$$

where the state transition matrices $\Psi(k, 0)$ and $\Psi(k, i)$ involve the $\underline{w}_i$ up to $\underline{w}_{k-1}$. The Ψ's may be computed from the relationships:

$$\Psi(i+1, i) = \Phi(i+1, i) [I - \underline{w}_i \underline{h}_i^T]$$

$$\Psi(i+1, i-1) = \Psi(i+1, i) \Psi(i, i-1)$$

$$\Psi(1, 0) = \Phi(1, 0)$$

where $\Phi(i+1, i)$ is the system state transition matrix.

In writing Eq. (6) it was assumed that there is no driving noise. The results to be obtained here would be the same if there were driving noise provided that the driving and measurement noises are uncorrelated.

Using Eqs. (2) and (6) plus the assumption that the initial estimation error is uncorrelated with the measurement noise for later times there results:

$$\underline{s}_k \triangleq \overline{\underline{e}'_k c_k} = \sum_{i=1}^{k-1} \Psi(k, i+1)\, \Phi(i+1, i)\, \underline{w}_i C_{ik} \tag{7}$$

Note that $\underline{s}_k$ is not a function of $\underline{w}_k$. Substitution of Eq. (7) into Eq. (6) yields:

$$P_k = (I-\underline{w}_k\underline{h}^T_k)P'_k\,(I-\underline{w}_k\underline{h}^T_k)^T + \underline{w}_k(C_{kk}+R_k)\,\underline{w}_k^T$$
$$+ (I-\underline{w}_k\underline{h}^T_k)\,\underline{s}_k\underline{w}_k^T + \underline{w}_k\underline{s}_k^T\,(I-\underline{w}_k\underline{h}^T_k)^T \tag{8}$$

Using Eq. (4) and the commutative property of conformable quantities inside the argument of the trace we obtain:

$$J = tr(P_k) = tr(P'_k) + 2(\underline{s}_k^T - \underline{h}_k^T P'_k)\,\underline{w}_k$$
$$+ (\underline{h}_k^T P'_k \underline{h}_k - 2\underline{h}_k^T \underline{s}_k + C_{kk} + R_k)\,\underline{w}_k^T \underline{w}_k$$

The condition for minimum mean squared error in the estimate is then:

$$\frac{\partial J}{\partial \underline{w}_k} = 0 = 2(\underline{s}_k - P'_k\underline{h}_k) + 2(\underline{h}_k^T P'_k \underline{h}_k - 2\underline{h}_k^T \underline{s}_k + C_{kk} + R_k)\underline{w}_k$$

Thus the optimum weighting vector is given by:

$$\underline{w}_k = \frac{1}{(\underline{h}_k^T P'_k\underline{h}_k - 2\underline{h}_k^T \underline{s}_k + C_{kk} + R)} \left[P'_k\underline{h}_k - \underline{s}_k \right] \tag{9}$$

Note that this reduces to the familiar Kalman gain vector when there is no colored measurement noise.

Equation (8) is used to update the covariance matrix at the k'th measurement time using $\underline{w}_k$ from Eq. (9). In most real time applications, computation of $\underline{w}_k$ and P_k by this method would only be practical if the Kalman gains were preprogrammed in the on board computer as they are in the LM digital autopilot.

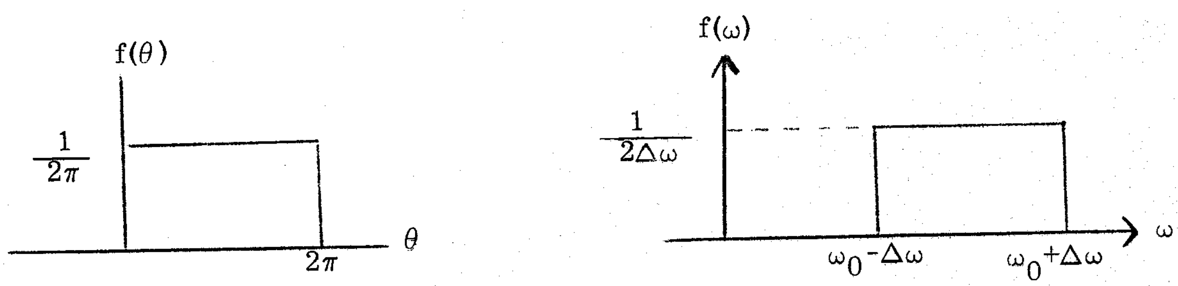
<u>Computation of $\underline{s}_k$ for a Random Sinusoid</u>

In this section we will obtain an expression for $\underline{s}_k$ in the special case where the correlated part of the measurement noise is a random sinusoid. This closely approximates the effect of slosh on the LM attitude measurements. If the additional assumptions of stationary statistics, known frequency, and periodic sampling at a specified rate are made, recursion formulas for $\underline{s}_k$ can be obtained. Specifically, we will assume:

$$c(t) = A \sin(\omega t + \theta)$$

where A, ω, and θ are independent random variables with the following statistics:

$$\overline{A^2} = \sigma^2$$



where $f(\theta)$ and $f(\omega)$ are probability density functions.

$$C_{ik} = \overline{c(t_i)\ c(t_k)} = \overline{A^2 \sin(\omega t_i + \theta)\ \sin(\omega t_k + \theta)}$$

where

$$t_k > t_i$$

Employing the independence property of A, ω and θ we may write, after some trigonometric manipulation:

$$C_{ik} = \overline{A^2}\ [\ \overline{\sin \omega t_i \sin \omega t_k}\ \overline{\cos^2\theta} + \overline{\cos\ \omega t_i\ \cos \omega t_k}\ \overline{\sin^2\theta}$$

$$+ \overline{\frac{1}{2} \sin 2\omega\ (t_i + t_k)}\ \overline{\sin 2\theta}\ ]$$

but with the assumed distribution of θ we have:

$$\overline{A^2} = \sigma^2$$

$$\overline{\cos^2\theta} = \overline{\sin^2\theta} = 1/2$$

$$\overline{\sin 2\theta} = \theta$$

thus:

$$C_{ik} = \frac{\sigma^2}{2}\ \overline{\cos \omega(t_i - t_k)} = \frac{\sigma^2}{2}\ \overline{\cos \omega(t_k - t_i)}$$

Using the statistical properties of ω to evaluate the remaining expectation there results:

$$C_{ik} = \frac{\sigma^2}{2\Delta\omega\ (t_k - t_i)} \sin \Delta\omega\ (t_k - t_i)\ \cos \omega_0\ (t_k - t_i) \qquad (10)$$

A limiting process when i=k yields:

$$C_{kk} = \frac{\sigma^2}{2} \tag{11}$$

Substitution of Eq. (10) into Eq. (7) yields the desired expression for $\underline{s}_k$:

$$\underline{s}_k = \sum_{i=1}^{k-1} \Psi(k, i+1)\ \Phi(i+1, i)\ \underline{w}_i \quad \frac{\sigma^2 \sin \Delta\omega (t_k - t_i) \cos \omega_0 (t_k - t_i)}{2\Delta\omega (t_k - t_i)} \tag{12}$$

If it is assumed that the statistics are stationary, the frequency is known ($\omega = \omega_0$), and that the sample period is constant (T) Eq. (10) becomes:

$$C_{kk} = \frac{\sigma^2}{2} \cos [\omega_0 T (k-i)]$$

then:

$$\underline{s}_k = \frac{\sigma^2}{2}\ \Psi(k, 1) \sum_{i=1}^{k-1} \Psi(1, i+1)\ \Phi(i+1, i)\ \underline{w}_i \cos[\omega_0 T(k-i)]$$

By defining an additional variable:

$$\underline{q}_k = \frac{\sigma^2}{2}\ \Psi(k, 1) \sum_{i=1}^{k-1} \Psi(1, i+1)\ \Phi(i+1, i)\ \underline{w}_i \sin[\omega_0 T(k-i)]$$

The following set of recursion formulas may be used to determine $\underline{s}_k$:

(13)

$$\underline{s}_{k+1} = \frac{\sigma^2}{2} \cos \omega_0 T\ \Phi(k+1, k)\underline{w}_k + \Psi(k+1, k) [\cos \omega_0 T\ \underline{s}_k - \sin \omega_0 T\ \underline{q}_k]$$

$$\underline{q}_{k+1} = \frac{\sigma^2}{2} \sin \omega_0 T\ \Phi(k+1, k)\underline{w}_k + \Psi(k+1, k) [\sin \omega_0 T\ \underline{s}_k + \cos \omega_0 T\ \underline{q}_k] \tag{14}$$

$$\underline{s}_1 = \underline{q}_1 = 0$$

Summary

A method for changing the Kalman gains to account for colored measurement noise when the dimension of the state is held fixed has been presented. Equation (9) together with Eq. (7) and the usual covariance matrix extrapolation equation are used to determine the new gains. The covariance matrix is updated according to Eq. (8). Equation (12), or in a more restrictive case Eqs. (13) and (14), can be used to compute $\underline{s}_k$ for those applications where the correlated noise is random sinusoid.