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Space Guidance Analysis Memo #3-66

TO: SGA Distribution

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DATE: January 14, 1966

SUBJECT: Derivation of the Optimum Control Program for Steering the LEM Using the Gimballed Descent Engine

Reference: Athans and Falb, "Optimal Control, An Introduction to the Theory and its Applications", (to be published)

1. Introduction

The attitude of the descent LEM is controlled by firing reaction jets and by gimballing the descent engine. Since there is no fuel penalty when using the gimbal, maximum use should be made of the gimbal in preference to the reaction jets. A control policy which might prove useful here is the time-optimal bang-bang control for the indicated triple-integral plant.

The basic differential equation relating a choice of gimbal drive u to the deviation of the vehicle attitude θ_v from the desired attitude θ_d is

$$\frac{d^3\theta}{dt^3} = \frac{FLR}{I} u \tag{1}$$

where $\theta = \theta_v - \theta_d$, F is the thrust, L is the distance from the engine gimbal axis to the vehicle center of mass, R is the gimbal drive constant angular rate, and I is the moment of inertia of the vehicle. The control signal u can take on the values +1, 0, or -1. These relationships are illustrated in Fig. 1. Note for Eq. (1) to be true the third derivative of the desired attitude must be zero.

We neglect the time variation of F, L, and I. Furthermore we assume that the deviation state variables $\ddot{\theta}$, $\dot{\theta}$, and θ are available to the controller with no measurement noise or estimation error. We can then derive the control program which drives the vehicle attitude into alignment with the desired attitude in the minimum possible time. It will be shown that this time optimal control program is the following memoryless nonlinear function of the present state:

$$\left.\begin{aligned}
c &= \left(\frac{I}{FLR}\right)^{1/3} \\
x_1 &= c^2 \frac{d^2\theta}{dt^2}, \qquad x_2 = c\,\frac{d\theta}{dt}, \qquad x_3 = \theta \\
s_1 &= \text{sign}\,(x_1) \\
s_2 &= \text{sign}\,(x_2 + \frac{1}{2}\, s_1\, x_1{}^2) \\
u_{opt} &= -\,\text{sign}\,[x_3 + \frac{1}{3}\, x_1{}^3 + s_2\, x_1\, x_2 + s_2\, (s_2\, x_2 + \frac{1}{2}\, x_1{}^2)^{3/2}]
\end{aligned}\right\} \quad (2)$$

This control program in theory will bring any initial state to the origin using no more than two reversals of the control signal. In practice a limit cycle exists whose amplitude depends on the choice of computer sample rate.

To simplify the notation in this memo, we re-write Eq. (1) in non-dimensional form. We define the non-dimensional time variable τ to be

$$\tau = \frac{t}{c} \qquad (3)$$

where

$$c = \left(\frac{I}{FLR}\right)^{1/3} \text{ sec.} \qquad (4)$$

We define the three non-dimensional state variables to be

$$x_1 = c^2 \frac{d^2\theta}{dt^2}, \qquad x_2 = c\,\frac{d\theta}{dt}, \qquad x_3 = \theta \qquad (5)$$

Then the non-dimension set of first order equations equivalent to Eq.(1) is

$$\left.\begin{aligned}
\dot{x}_1 &= u \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= x_2
\end{aligned}\right\} \qquad (6)$$

where the dot indicates differentiation with respect to τ. This equivalent system is illustrated in Fig. 2.

2. A Necessary Condition for Optimality

We wish to minimize the cost functional

$$T = \int_0^T 1 \, d\tau \tag{7}$$

The Hamiltonian is the sum of the integrand in Eq. (7) plus the adjoined differential Eqs. (6).

$$\begin{aligned} H &= L + \langle \underline{p}, \ \dot{\underline{x}} \rangle \\ H &= 1 + p_1 u + p_2 x_1 + p_3 x_2 \end{aligned} \tag{8}$$

The time derivative of the adjoint vector $\underline{p}$ is the negative of the gradient of the Hamiltonian with respect to the state variables.

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{x}} \Rightarrow \left.\begin{aligned} \dot{p}_3 &= 0 \\ \dot{p}_2 &= -p_3 \\ \dot{p}_1 &= -p_2 \end{aligned}\right\} \tag{9}$$

The general solution of the adjoint Eq. (9) is

$$\left.\begin{aligned} p_3 &= 2 K_2 \\ p_2 &= -K_1 - 2K_2 \tau \\ p_1 &= K_0 + K_1 \tau + K_2 \tau^2 \end{aligned}\right\} \tag{10}$$

Pontryagin's minimum principle states: a necessary condition that a choice of control u be optimum is that the Hamiltonian H be minimized by that choice. We have +1, 0, and -1 as admissible control signals. The choice which minimizes H is

$$u = - \text{sign} \, [p_1] \tag{11}$$

Considering Eq. (10) we see that p_1 can change signs never, once, or twice depending on the constants. We conclude that the only control sequences that could possibly be optimal are

$$(+1),\ (-1),\ (+1,-1),\ (-1,\ +1),\ (+1,-1,\ +1),\ (-1,\ +1,\ -1) \tag{12}$$

A possible exception to conclusion 12 could occur if $p_1(\tau)$ were zero over some finite interval. In such an interval condition 11 would not determine u. But this will not happen because from Eq. (10), $p_1(\tau) = 0$ implies $K_0 = K_1 = K_2 = 0$, which in turn implies p_2 and p_3 are also zero. And from Eq. (8), the Hamiltonian must therefore equal unity. This is a contradiction because the Hamiltonian is known to be zero along an optimal trajectory of a free terminal time problem.

Thus in searching for the optimal control program, we need consider only those programs which will bring any initial state (x_1, x_2, x_3) to the origin with no more than two reversals of the control signal. Our derivation of the optimal control program proceeds by constructing one such program. Then the program so constructed is proven to be the only one that exists which meets the minimum switching condition. Since it is unique it must be the optimum.

3. Derivation of One Control Program

Define V_1^+ to be the set of all states from which the origin can be reached in positive time by applying the single terminal control $u = +1$. Similarly define V_1^- to be the set of all states from which the origin can be reached in positive time by applying the single terminal control $u = -1$. Further we define V_1 to be the union of set V_1^+ and set V_1^-. That is V_1 is the set of all states from which the origin can be reached in positive time by applying either $u = +1$ or $u = -1$.

We shall develop an alternate coordinate system for the state space, which is more convenient than x_1, x_2, x_3 for proving properties. The first of these coordinates d_1 is defined as

$$d_1 = x_1 \tag{13}$$

An important property of the set V_1 is that states in V_1 may be assigned uniquely to V_1^+ or V_1^- according to the sign of the coordinate d_1. We state this as

PROPERTY 1:

If $\underline{x} \in V_1$ and $d_1 > 0$, then $\underline{x} \in V_1^-$

If $\underline{x} \in V_1$ and $d_1 < 0$, then $\underline{x} \in V_1^+$

To prove this property, we compute the time derivative of d_1 using Eq. (13) and (6)

$$\dot{d}_1 = u \tag{14}$$

It is clear if the control is held constant at $u = +1$ and if d_1 is initially positive that d_1 will never be driven to zero. This is illustrated in Fig. 3. Since the origin is a state contained in the plane $d_1 = 0$, the origin will never be reached. Hence if d_1 is positive the state can not be in V_1^+. But if the state is known to be in V_1 then it must be in V_1^-. The proof of the second statement in property 1 is the same.

We may use property 1 as the basis of a control policy to be applied whenever a state is in the set V_1.

CONTROL POLICY 1

If $\underline{x} \in V_1$, then apply $u = -\operatorname{sign}(d_1)$

We need an explicit formula for the set V_1. We begin by noting that the general solution to the system of Eq. (6), in an interval where the control signal is held constant, is

$$\left.\begin{aligned} x_1 &= c_1 + u\tau \\ x_2 &= c_2 + c_1\tau + \frac{1}{2}u\tau^2 \\ x_3 &= c_3 + c_2\tau + \frac{1}{2}c_1\tau^2 + \frac{1}{6}u\tau^3 \end{aligned}\right\} \tag{15}$$

The set of all states from which the state $\underline{c}$ can be reached in positive time by applying the constant control u may be generated as

$$\left.\begin{aligned} x_1 &= c_1 - ub \\ x_2 &= c_2 - c_1 b + \frac{1}{2}ub^2 \\ x_3 &= c_3 - c_2 b + \frac{1}{2}c_1 b^2 - \frac{1}{6}ub^3 \end{aligned}\right\} \quad 0 < b < \infty \tag{16}$$

In particular the set V_1 is generated by

$$\left.\begin{aligned} x_1 &= - u_1 b \\ x_2 &= \frac{1}{2} u_1 b^2 \\ x_3 &= - \frac{1}{6} u_1 b^3 \end{aligned}\right\} \quad \begin{aligned} &0 < b < \infty \\ &u_1 = \pm 1 \end{aligned} \tag{17}$$

If we use the first formula in set 17 to eliminate b from the second and third, we obtain for the set V_1

$$\left.\begin{aligned} x_2 &= \frac{1}{2} u_1 x_1^2 \\ x_3 &= \frac{1}{6} x_1^3 \end{aligned}\right\} \quad \begin{aligned} &-\infty < x_1 < \infty \\ &u_1 = \pm 1 \end{aligned} \tag{18}$$

And finally, we use property 1 (or control policy 1) to replace u_1 with an explicit function of the coordinate d_1. This gives us for the set V_1

$$s_1 = \text{sign}\,(d_1) \tag{19}$$

$$\left.\begin{aligned} x_2 &= - \frac{1}{2} s_1 x_1^2 \\ x_3 &= \frac{1}{6} x_1^3 \end{aligned}\right\} \quad -\infty < x_1 < \infty \tag{20}$$

It is clear that the set V_1 is a curve in our three-dimensional state space.

Define V_2^- to be the set of all states from which the origin can be reached in positive time by applying a control sequence (-1, +1). It is clear that the set V_2^- may be found by generating the set of all states from which some state in the set V_1^+ can be reached in positive time by applying the control u = -1. Similarly define V_2^+ to be the set of all states from which the origin can be reached in positive time by applying a control sequence (+1, -1). It is clear that the set V_2^+ may be found by generating the set of all states from which some state in the set V_1^- can be reached in positive time by applying the control u = +1. We define V_2 to be the union of set V_2^- and set V_2^+. That is V_2 is the set of all states from which the origin can be reached in positive time by applying either a control sequence (-1, +1) or a control sequence (+1, -1).

We introduce a second coordinate d_2 which is a measure of the distance of a state from the curve V_1

$$d_2 = x_2 + \frac{1}{2} s_1 x_1^2 \tag{21}$$

We note from Eq. (20) that a state can be in the curve V_1 only if $d_2 = 0$.

An important property of the set V_2 is that states in V_2 may be assigned uniquely to V_2^- or V_2^+ according to the sign of the coordinate d_2. We state this as

PROPERTY 2:

If $\underline{x} \in V_2$ and $d_2 > 0$, then $\underline{x} \in V_2^-$

If $\underline{x} \in V_2$ and $d_2 < 0$, then $\underline{x} \in V_2^+$

To prove this property, we compute the time derivative of d_2 using definitions 21 and 13 with Eqs. (6).

$$\dot{d}_2 = (1 + s_1 u) d_1 \tag{22}$$

Suppose we apply the control $u = +1$. Equation (22) then shows that where d_1 is negative $\dot{d}_2$ is zero and where d_1 is positive $\dot{d}_2$ is positive. Hence nowhere is $\dot{d}_2$ negative. Hence no state in a region where d_2 is positive can be driven into the surface $d_2 = 0$ by applying the control $u = +1$. This statement is illustrated in Fig. 4. Now since the curve V_1 is in the surface $d_2 = 0$, it is also true that V_1 can not be reached. Hence if a state is where d_2 is positive the state can not be in V_2^+. But if the state is known to be in V_2 then it must be in V_2^-, which proves that first half of property 2. The proof of the second half is the same.

We may use property 2 as the basis of a control policy to be applied whenever a state is in the set V_2

CONTROL POLICY 2

If $\underline{x} \in V_2$, then apply $u = -\text{sign}(d_2)$

We now wish to develop an explicit formula for the set V_2. In generating relations 16 we specify that $\underline{c}$ is in V_1 and u is given explicitly by property 2 (or control policy 2). Thus

$$s_2 = \text{sign}\,(d_2) \tag{23}$$

$$\left.\begin{aligned} x_1 &= c_1 + s_2 b \\ x_2 &= c_2 - c_1 b - \tfrac{1}{2} s_2 b^2 \\ x_3 &= c_3 - c_2 b + \tfrac{1}{2} c_1 b^2 + \tfrac{1}{6} s_2 b^3 \end{aligned}\right\} \quad \begin{aligned} &0 < b < \infty \\ &\underline{c} \in V_1 \end{aligned} \tag{24}$$

Since $\underline{c}$ is in V_1 we may use formula 20 to eliminate c_2 and c_3 from Eq. (24). In formula 20, s_1 is the sign of d_1 (which is c_1). If s_2 is positive, then we know negative control is required to reach V_1. We will intersect the curve V_1 at a c_1 which is in V_1^{+}. And in V_1^{+} s_1 is negative. Similarly s_2 negative at $\underline{x}$ implies s_1 positive at $\underline{c}$. So use $s_1 = - s_2$. The set V_2 is now expressed as

$$\left.\begin{aligned} x_1 &= c_1 + s_2 b \\ x_2 &= \tfrac{1}{2} s_2 c_1^{2} - c_1 b - \tfrac{1}{2} s_2 b^2 \\ x_3 &= \tfrac{1}{6} c_1^{3} - \tfrac{1}{2} s_2 c_1^{2} b + \tfrac{1}{2} c_1 b^2 + \tfrac{1}{6} s_2 b^3 \end{aligned}\right\} \quad \begin{aligned} &0 < b < \infty \\ &-\infty < c_1 < \infty \end{aligned} \tag{25}$$

Use the first formula of relations 25 to eliminate the generating variable b from the second and third formulas. The set V_2 is then written as

$$\left.\begin{aligned} x_2 &= s_2 c_1^{2} - \tfrac{1}{2} s_2 x_1^{2} \\ x_3 &= c_1^{3} - c_1^{2} x_1 + \tfrac{1}{6} x_1^{3} \end{aligned}\right\} \quad \begin{aligned} &-\infty < c_1 < \infty \\ &s_2 c_1 < s_2 x_1 < \infty \end{aligned} \tag{26}$$

The first formula of 26 can be solved for the point c_1 where a state in V_2 would be driven into V_1

$$c_1 = - s_2 \left(s_2 x_2 + \tfrac{1}{2} x_1^{2}\right)^{1/2} \tag{27}$$

The sign of the root in formula 27 has been set to $-s_2$. The reason for this choice is, as mentioned before, that in regions of V_2 where s_2 is negative (that is in V_2^+) the state is driven to the set V_1^- where c_1 is positive. And in regions of V_2 where s_2 is positive the state is driven into V_1^+ where c_1 is negative. We use formula 27 to eliminate c_1 from the second formula of 26. This gives us our desired explicit formula for the set V_2 which is valid both in V_2^- and V_2^+.

$$x_3 = -\frac{1}{3} x_1^3 - s_2 x_1 x_2 - s_2 (s_2 x_2 + \frac{1}{2} x_1^2)^{3/2} \tag{28}$$

It is clear that, whereas V_1 was a curve, V_2 is a surface in our three dimensional space.

Define V_3^+ to be the set of all states from which the origin can be reached in positive time by applying a control sequence (+1, -1, +1). Similarly define V_3^- to be the set of all states from which the origin can be reached in positive time by applying a control sequence (-1, +1, -1). We define V_3 to be the union of set V_3^+ and V_3^-. That is V_3 is the set of all states from which the origin can be reached in positive time by applying either a control sequence (+1, -1, +1) or a control sequence (-1, +1, -1).

We introduce our third coordinate d_3 which is a measure of the distance of a state from the surface V_2

$$d_3 = x_3 + \frac{1}{3} x_1^3 + s_2 x_1 x_2 + s_2 (s_2 x_2 + \frac{1}{2} x_1^2)^{3/2} \tag{29}$$

We note from Eq. (28) that a state is in the surface V_2 if and only if $d_3 = 0$.

An important property of the set V_3 is that states in V_3 may be assigned uniquely to V_3^+ or V_3^- according to the sign of the coordinate d_3. We state this as

PROPERTY 3

If $\underline{x} \in V_3$ and $d_3 > 0$, then $\underline{x} \in V_3^-$

If $\underline{x} \in V_3$ and $d_3 < 0$, then $\underline{x} \in V_3^+$

The proof again follows from the time derivative of the relevent coordinate. Using the coordinate definitions 29, 21, and 13 with the system differential equations 6 we find

$$\dot{d}_3 = (1 + u\, s_2)\ \{d_2 + (2s_2 - s_1) \frac{1}{2} d_1^2 + \frac{3}{2} d_1 [s_2 d_2 + (\frac{1 - s_1 s_2}{2}) d_1^2]^{1/2}\} \tag{30}$$

Examining formula 30 in each of four regions of our state space we find

Region I: $d_1 > 0$ $(s_1 = +1)$, $d_2 > 0$ $(s_2 = +1)$

$$\dot{d}_3 = (1 + u)\ \underbrace{(d_2 + \tfrac{1}{2} d_1^{\ 2} + \tfrac{3}{2} d_1 \sqrt{d_2})}_{\text{positive}} \qquad \geq 0 \tag{31}$$

Region II: $d_1 < 0$ $(s_1 = -1)$, $d_2 > 0$ $(s_2 = +1)$

$$\dot{d}_3 = (1 + u)\ \underbrace{(d_2 + \tfrac{3}{2} d_1^{\ 2} + \tfrac{3}{2} d_1 \sqrt{d_2 + d_1^{\ 2}}\ )}_{\text{positive}} \qquad \geq 0 \tag{32}$$

Region III: $d_1 < 0$ $(s_1 = -1)$, $d_2 < 0$ $(s_2 = -1)$

$$\dot{d}_3 = (1 - u)\ \underbrace{(d_2 - \tfrac{1}{2} d_1^{\ 2} + \tfrac{3}{2} d_1 \sqrt{s_2 d_2}\ )}_{\text{negative}} \qquad \leq 0 \tag{33}$$

Region IV: $d_1 > 0$ $(s_1 = +1)$, $d_2 < 0$ $(s_2 = -1)$

$$\dot{d}_3 = (1 - u)\ \underbrace{(d_2 - \tfrac{3}{2} d_1^{\ 2} + \tfrac{3}{2} d_1 \sqrt{s_2 d_2 + d_1^{\ 2}})}_{\text{negative}} \qquad \leq 0 \tag{34}$$

In region I it was obvious that the factor in brackets is positive because each term in that factor is positive. It is less obvious in the case of region II. To prove this in region II we note that the first two terms in the factor are positive while the third term is negative. We can compare the relative magnitude of the positive terms versus the negative term by squaring both groups

$$(d_2 + \tfrac{3}{2} d_1^{\ 2})^2 \ \text{vs.}\ (\tfrac{3}{2} d_1 \sqrt{d_2 + d_1^{\ 2}})^2 \tag{35}$$

$$d_2^{\ 2} + \tfrac{12}{4}\ d_1^{\ 2} d_2 + \tfrac{9}{4} d_1^{\ 4} \ \text{vs.}\ \tfrac{9}{4} d_1^{\ 2} d_2 + \tfrac{9}{4} d_1^{\ 4} \tag{36}$$

Result 36 shows that the positive terms have a larger magnitude than the negative term, which proves our assertion that the factor in brackets for region II is positive. Similarly in region III it was obvious that the factor was negative. And, in region IV a comparison of magnitudes proves the factor negative. Now suppose we apply the control $u = +1$. Equations(31) to(34)show that in the four regions $\dot{d}_3$ is positive, positive, zero, and zero respectively. Since these four regions include all of the state space we conclude that for u = + 1 nowhere can $\dot{d}_3$ be negative. Hence no state in a region where d_3 is positive can be driven to the surface $d_3 = 0$ by applying the control u = + 1. And since $d_3 = 0$ is by definition the surface V_2 the same statement applies with respect to V_2. Thus such a state cannot be in V_3^+. But if the state is known to be in V_3 then it must be in V_3^-, which proves the first statement of property 3. The proof of the second statement is identical.

We may use property 3 as the basis of a control policy to be applied when ever a state is in the set V_3.

CONTROL POLICY 3

If $\underline{x} \in V_3$, then apply $u = -\operatorname{sign}(d_3)$

We have developed three control policies one of which may be used provided we are at a state in V_1, or V_2 or V_3. We now claim that every state in our state space is contained in one of these three sets of states. That is for every initial state there exists a way of driving the state to the origin with no more than two control reversals. First we note that the coordinate transformation yielding d_1, d_2, and d_3 (see definitions 13, 21, and 29) is defined for all states $\underline{x}$. Now if d_3 is not equal to zero, does that necessarily imply that a state is contained in the set V_3? We already proved that if $d_3 > 0$ then V_2 cannot be reached by applying the constant control u = + 1. So we must prove that if we try u = - 1 V_2 will be reached without fail. We note from formula 14 that for u = - 1, $\dot{d}_1$ is always negative. Thus d_1 will eventually become negative and will remain negative. We see from formula 22 that $\dot{d}_2$ becomes negative as soon as d_1 goes negative. Thus d_2 will eventually become negative and will remain negative. And finally from formula 33, since d_1 and d_2 are both negative and the control is negative, $\dot{d}_3$ is negative. Thus, any initial state where d_3 was positive will be driven into the surface $d_3 = 0$ by applying u = - 1. Similarly we can prove that where d_3 is negative, applying u = + 1 will drive any state into $d_3 = 0$. Thus we have proved that all states where d_3 is not zero are contained in the set V_3. In the cases where $d_3 = 0$ we are by definition on the surface V_2 or the curve V_1. But if d_2 is not zero we cannot be in the curve V_1.

The above discussion provides a unique method for assigning any state to one of the sets V_3, V_2, or V_1 according to the coordinates d_3, d_2, and d_1. We combine this assignment method with the three control policies already stated to give an explicit control program, which meets the necessary condition that any initial state is transferred to the origin with no more than two control reversals:

EXPLICIT CONTROL PROGRAM

$$\left.\begin{array}{l}
d_1 = x_1 \\
s_1 = \text{sign}\,(d_1) \\
d_2 = x_2 + \frac{1}{2} s_1 x_1^{\,2} \\
s_2 = \text{sign}\,(d_2) \\
d_3 = x_3 + \frac{1}{3} x_1^{\,3} + s_2 x_1 x_2 + s_2 (s_2 x_2 + \frac{1}{2} x_1^{\,2})^{3/2} \\
s_3 = \text{sign}\,(d_3) \\
\text{If } d_3 \neq 0, \qquad\qquad\qquad \text{apply } u = -s_3 \\
\text{If } d_3 = 0, \ \text{If } d_2 \neq 0, \quad \text{apply } u = -s_2 \\
\text{If } d_3 = 0, \ \text{If } d_2 = 0, \quad \text{apply } u = -s_1
\end{array}\right\} \tag{37}$$

4. Proof that the Control Program is Uniquely Optimal

We know that control policy 3 will drive any state in V_3 to V_2. We also proved that the opposite policy could never reach V_2. Thus if at any point in V_3 we deviate from policy 3 and apply the opposite control, we will never reach V_2 unless we reverse the control and return to policy 3. But introducing a reversal in V_3 necessarily implies that the origin will not be reached in a way that meets the minimum switching condition. Thus any alternate policy in V_3 cannot be an optimal policy.

On the surface V_2 we have $d_3 = 0$ and $d_2 \neq 0$. We know that control policy 2 will drive any state in V_2 to V_1. But suppose at some point we deviate from policy 2 and apply the opposite control. In particular suppose we apply $u = +1$ where $d_2 > 0$. We see from formulas 31 and 32 that $\dot{d}_3$ instantly becomes positive. So the state moves into a region of V_3 where $d_3 > 0$. And with $u = +1$ and $d_3 > 0$ the surface V_2 will not be reached again unless we reverse the control and apply policy 3. But introducing a reversal in V_3 necessarily implies that the minimum switching condition cannot be met. Thus any alternate policy to policy 2 in V_2 cannot be an optimal policy.

In the curve V_1 we have $d_3 = 0$ and $d_2 = 0$. We know that control policy 1 will drive any state in V_1 to the origin. But suppose at some point we deviate from policy 1 and apply the opposite control. In particular suppose we apply $u = +1$ where $d_1 > 0$. We see from formula 22 that $\dot{d}_2$ instantly becomes positive, so d_2 is driven positive. With $u = +1$, and both d_1 and d_2 positive formula 31 shows that $\dot{d}_3$ is positive. Again we are in region V_3 and a reversal is required if we are ever to reach V_2. Thus any alternative to policy 1 in V_1 cannot be an optimal policy.

Now since we have proven that no other policies exist, which can drive any state to the origin with no more than two control reversals, we have proven that our control program is uniquely optimal.

A typical trajectory driven by the optimal control program is shown in Fig. 5. The corresponding waveforms are shown in Fig. 6.

5. Some Engineering Considerations

In a practical application it is impossible to time a control reversal so that a trajectory is transferred precisely onto the surface V_2. For example in a computer control program the computation delay, between the discovery that d_3 has passed zero and the commanding of the required control reversal, allows the trajectory to integrate a finite distance past the switching surface. Even if the improbable occurred and the control was reversed precisely on the analytic surface V_2, the ensuing trajectory would soon wander off the analytic surface, because the actual vehicle is not perfectly described by the assumed differential equation. Therefore we need to program only policy 3. This policy is adequate to bring any state sufficiently close to the origin.

Suppose the control program is synthesized by a computer which only samples the deviation state with a uniform sampling rate $\Delta\tau$ and only makes decisions to reverse the control signal at these instants. In this case, after the initial transient dies out, the process will exhibit a limit cycle. At one sample time d_3 is found positive so negative control is commanded. At the next sample time d_3 has been driven negative so positive control is commanded. And so forth. We may compute the amplitude of this limit cycle as a function of our choice of sample period $\Delta\tau$. Suppose d_3 was found negative at a state $\underline{c}$ and positive control has been commanded. From relations 15 the ensuing trajectory will be

$$\left.\begin{aligned} x_1 &= c_1 + \tau \\ x_2 &= c_2 + c_1\tau + \frac{1}{2}\tau^2 \\ x_3 &= c_3 + c_2\tau + \frac{1}{2}c_1\tau^2 + \frac{1}{6}\tau^3 \end{aligned}\right\} \quad 0 < \tau < \Delta\tau \tag{38}$$

When $\tau = \Delta\tau$, if we are in the limit cycle, the state will have reached a point symmetrically opposite the initial point. That is

$$\left.\begin{aligned} -c_1 &= c_1 + \Delta\tau \\ -c_2 &= c_2 + c_1\Delta\tau + \frac{1}{2}(\Delta\tau)^2 \\ -c_3 &= c_3 + c_2\Delta\tau + \frac{1}{2}c_1(\Delta\tau)^2 + \frac{1}{6}(\Delta\tau)^3 \end{aligned}\right\} \tag{39}$$

The solution of the set of Eqs. 39 is

$$c_1 = -\left(\frac{\Delta\tau}{2}\right), \qquad c_2 = 0, \qquad c_3 = \frac{1}{3}\left(\frac{\Delta\tau}{2}\right)^3 \tag{40}$$

The trajectories in this limit cycle are shown in Fig. 7. It is easily shown that the maximum excursion of each component of the state vector is

$$x_1 = \frac{\Delta\tau}{2}, \qquad x_2 = \frac{1}{2}\left(\frac{\Delta\tau}{2}\right)^2, \qquad x_3 = \frac{1}{3}\left(\frac{\Delta\tau}{2}\right)^3 \tag{41}$$

If we use definitions 3, 4, and 5 we return to the original state variables. The maximum values are then given by

$$K = \frac{I}{FLR}\ \text{sec.}^3 \tag{42}$$

$$\ddot{\theta} = \frac{1}{K}\left(\frac{\Delta t}{2}\right), \qquad \dot{\theta} = \frac{1}{K}\frac{1}{2}\left(\frac{\Delta t}{2}\right)^2, \qquad \theta = \frac{1}{K}\frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \tag{43}$$

For an illustration, consider the descent configuration of the lunar excursion module at minimum inertia and maximum thrust. The numbers in this case give a value for the inverse of K of about $K^{-1} = 1$ degree/sec^3. If we choose to exercise the gimbal control program every 2 seconds, then the resulting limit cycle will have a maximum acceleration of 1 deg/sec^2, a maximum rate of 1/2 deg. /sec, and a maximum deviation of 1/3 degree.

By considering the size of the limit cycle we also have a basis for choosing a dead-zone for the reaction control system. Namely, if the objective of minimizing the use of RCS fuel is to be met, then certainly no additional control action should be taken by the RCS control program if the state lies in the vicinity of the origin as defined by the magnitude of the limit cycle. Only if the vehicle takes a large excursion away from this region should RCS impulses be used to help bring the state back to the origin.

The required control program was stated in summary 2. If the indicated cube-root in the first formula is wasteful of computer time, then the set may be reformulated in terms of $K = c^3$. This results in the following equivalent control program

$$\left.\begin{array}{l}
K = \dfrac{I}{FLR} \\
\ddot{\theta} = \dfrac{d^2\theta}{dt^2}, \qquad \dot{\theta} = \dfrac{d\theta}{dt}, \qquad \theta = \theta \\
s_1 = \text{sign}\,(\ddot{\theta}) \\
s_2 = \text{sign}\,(\dot{\theta} + \frac{1}{2}s_1 K \ddot{\theta}^2) \\
u = -\text{sign}\,[\theta + \frac{1}{3}K^2\ddot{\theta}^3 + s_2 K \ddot{\theta}\dot{\theta} + s_2 K^2 (s_2 \frac{1}{K}\dot{\theta} + \frac{1}{2}\ddot{\theta}^2)^{3/2}]
\end{array}\right\} \tag{44}$$

Fig. 1 - The assumed dynamics of the desired attitude θ_d and the vehicle attitude θ_v.

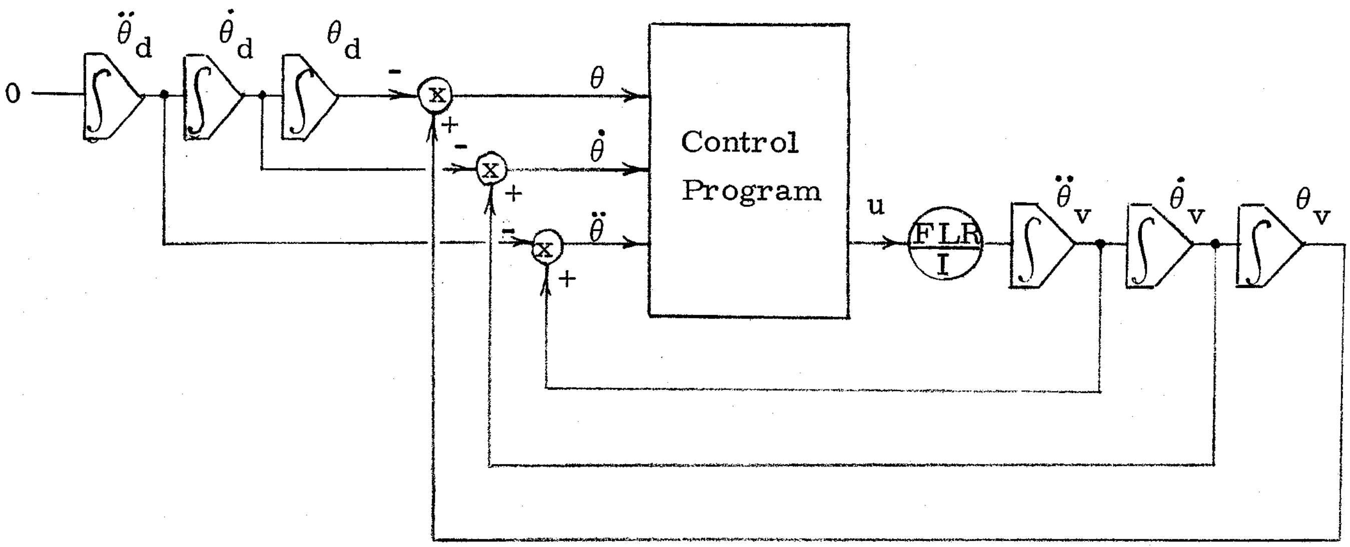

Fig. 2 - An equivalent reduced non-dimensional problem

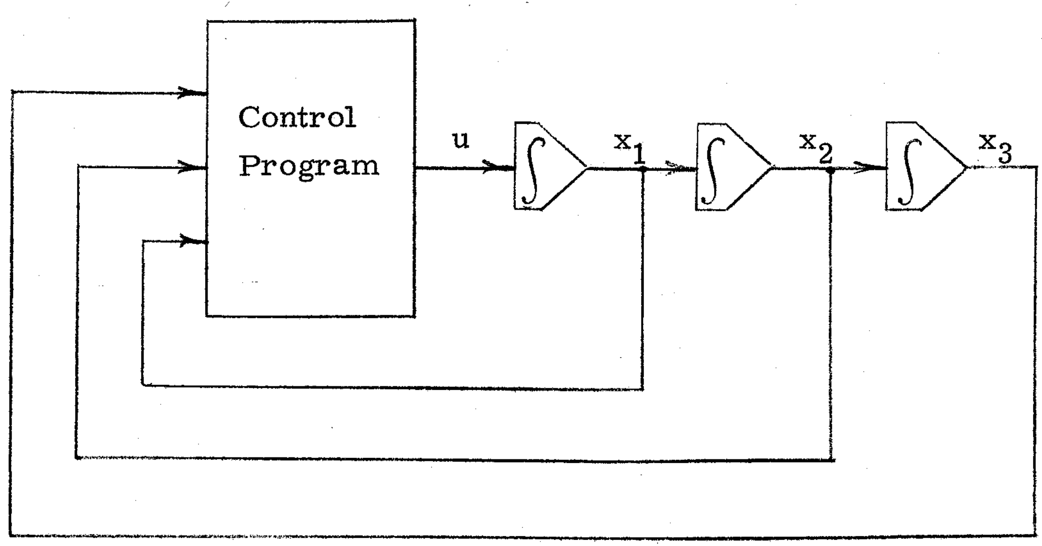

Fig. 3 - No state can be driven to the plane $d_1 = 0$ by applying the constant control $u = \text{sign}(d_1)$

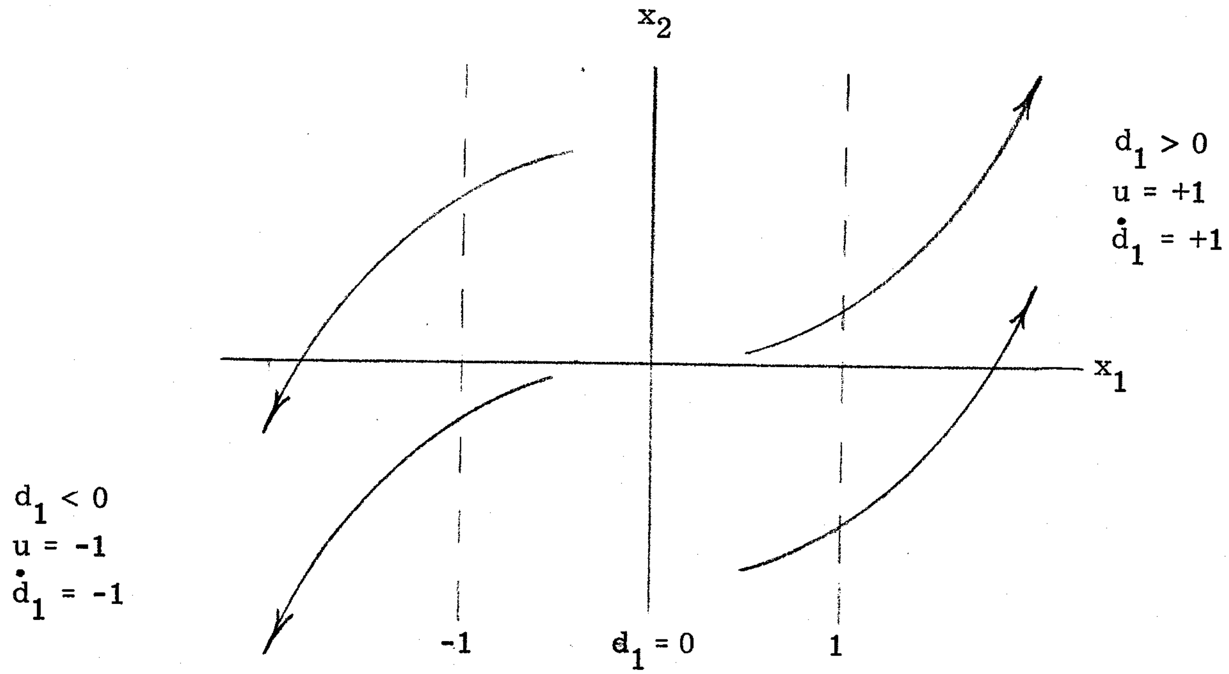

Fig. 4 - No state can be driven to the surface $d_2 = 0$ by applying the constant control $u = \text{sign}(d_2)$

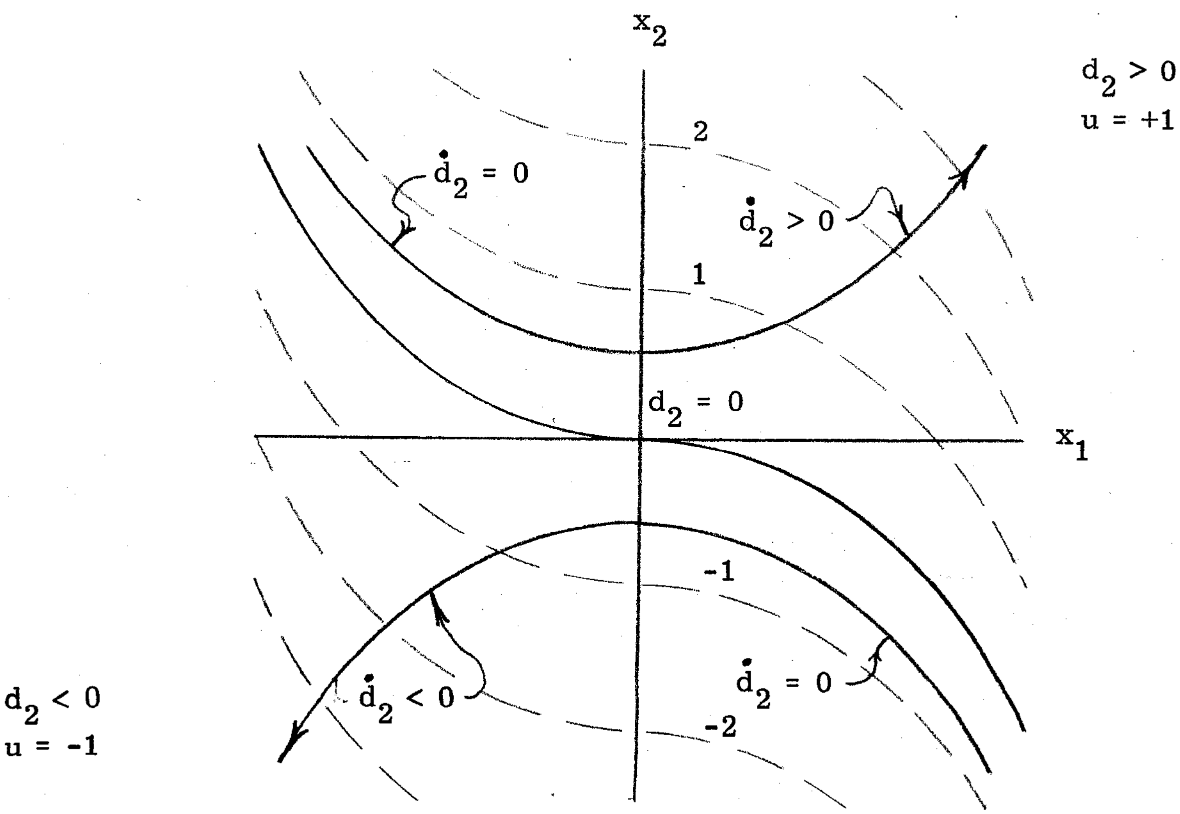

Fig. 5 - Minimum time recovery from an initial bias acceleration – trajectory in state space.

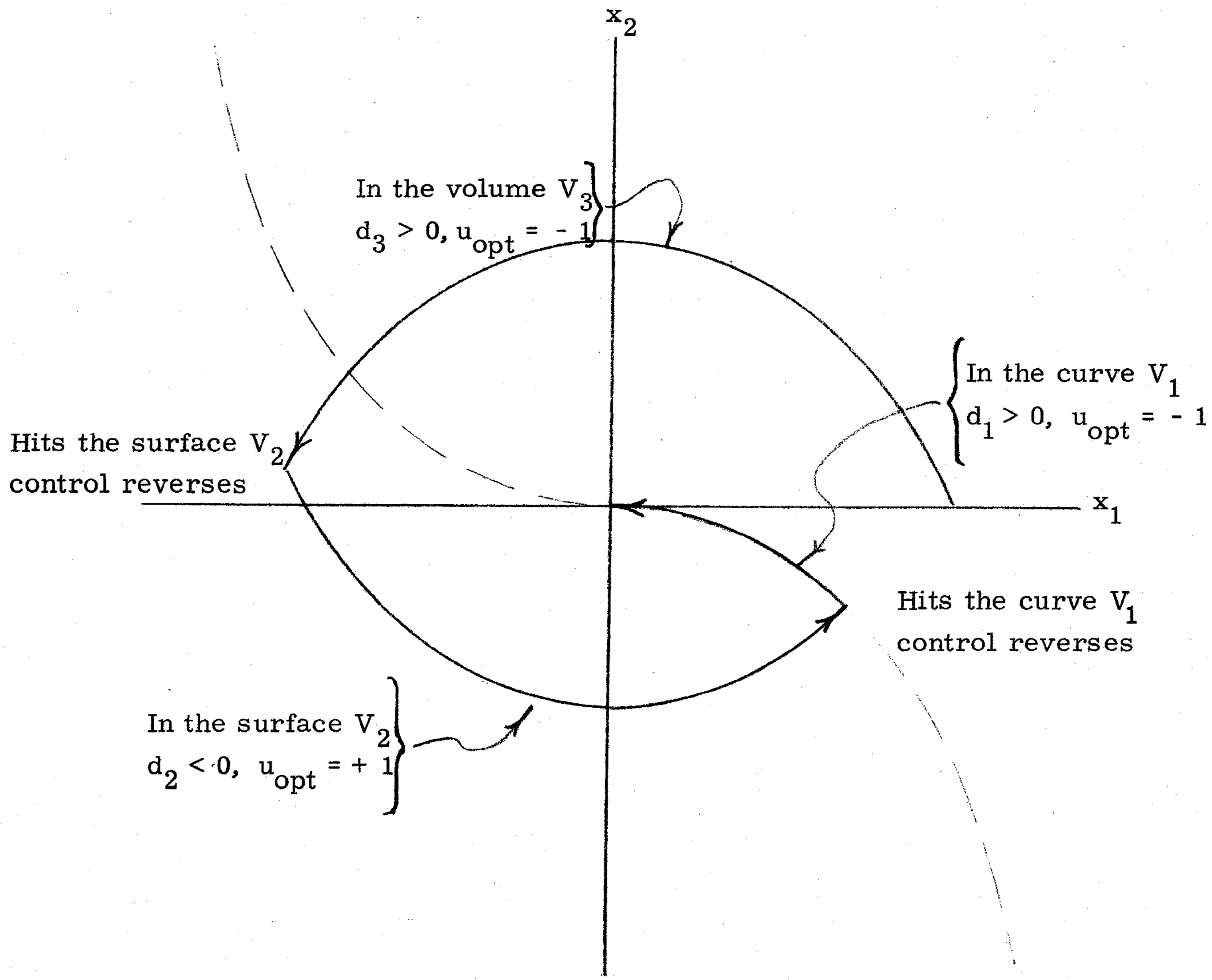

Fig. 6 - Minimum time recovery from an initial bias acceleration – waveforms in the time domain.

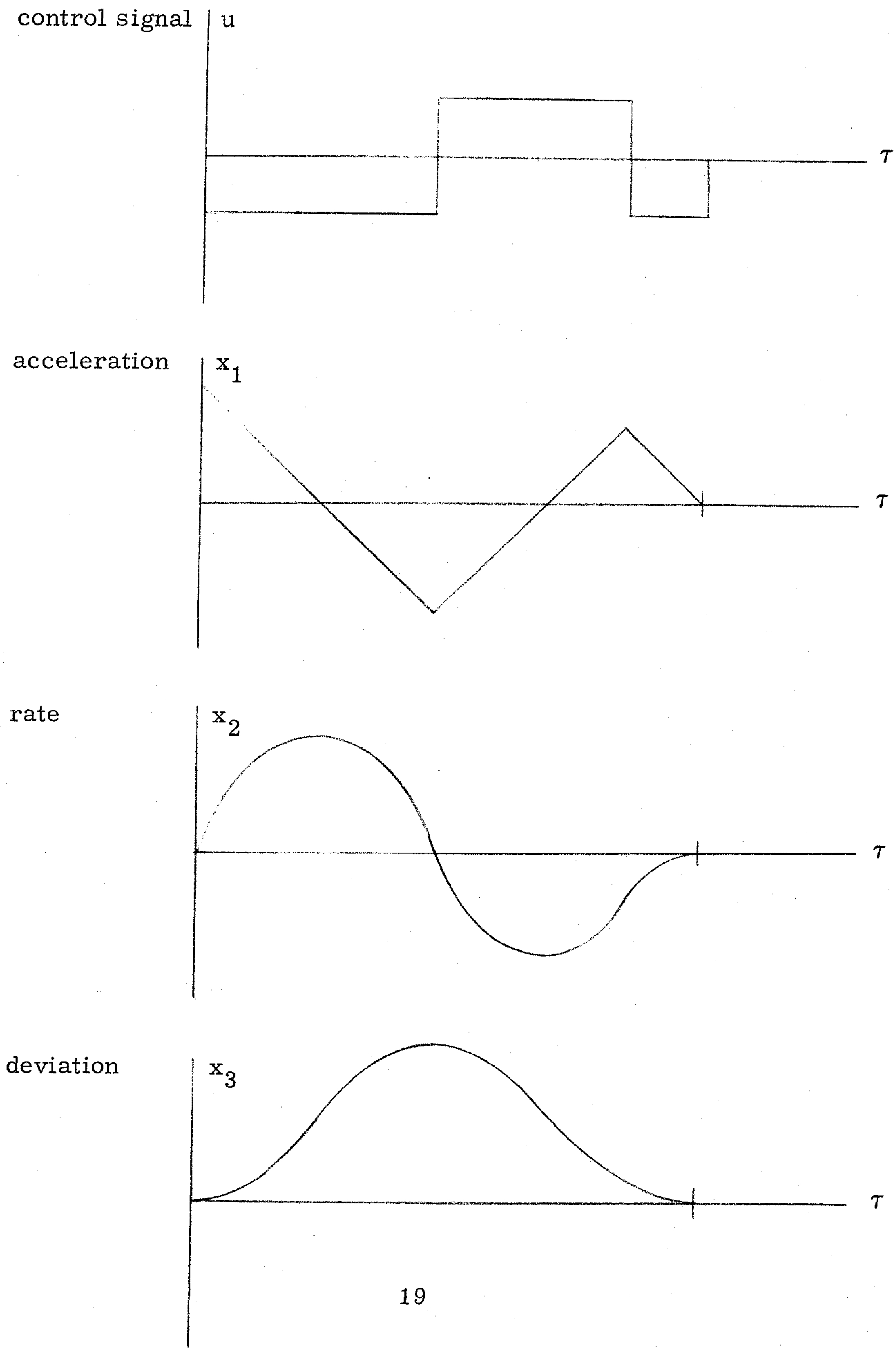

Fig. 7 - Limit cycle due to sampled data.