Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

Space Guidance Analysis Memo # 3-65

TO:	SGA Distribution
FROM:	Wayne Tempelman
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SUBJECT:	A Generalized Solution to the Impulsive Guidance Problem

Summary

A general solution is obtained to the impulsive guidance problem valid for constraints expressed as functions of the perturbations in the initial and final state vectors and in the time of flight. Two special problems are discussed to illustrate the variety of guidance laws which may be solved with the general solution. One is a reentry problem where the constraints are that the reentry angle, altitude and velocity are maintained while minimizing the distance from the nominal reentry point. The other is an entry into circular orbit problem where the nominal flight time, the impulse to enter the circular orbit and the plane normal to the orbit containing the terminal position vector are maintained while minimizing the velocity correction.

I. Introduction

Impulsive guidance corrections serve to nullify the effects of initial position and velocity errors under specified mission constraints. Examples are fixed and variable arrival time guidance laws. The mission constraints can be generalized and a general solution obtained to the impulsive guidance problem.

The specification of the nominal trajectory and the initial position perturbation leaves three conditions to be determined to define a new trajectory and one condition to determine the terminal point. There are thus four constraints which must be imposed. Two situations will be considered: the case of four independent constraints and the case of minimizing a specified function while satisfying three independent constraints.

II. Four Independent Constraints

Four independent scalar constraints can be imposed which are functions of the perturbations in the initial (subscript i) and final (no subscript) state vectors and in the time of flight (perturbations in other variables can be converted to perturbations in time of flight).

$$A\delta R_i + B\delta V_i + C\delta R + D\delta V + E\,\delta t = [0] \tag{1}$$

where A, B, C and D are 4×3 guidance constraint matrices, E is a 4×1 guidance constraint vector, $[0]$ is a null vector and δV_i is the velocity error after the correction δV_c has been applied ($\delta V_c = \delta V_i - \delta V_{initial}$). The final state error is related to the initial state error through the (partitioned) time transition matrix P. *

$$\begin{aligned} \delta R &= P_1\,\delta R_i + P_2\,\delta V_i + V_r\,\delta t \\ \delta V &= P_3\,\delta R_i + P_4\,\delta V_i + A_r\,\delta t \end{aligned} \tag{2}$$

where V_r and A_r are the differences in the velocity and acceleration between the nominal and target trajectories at the terminal point. If there is no target trajectory V_r and A_r become the velocity V_n and acceleration A_n of the nominal trajectory. Inserting Eq. (2) into (1) results in

$$[A + CP_1 + DP_3]\,\delta R_i + [B + CP_2 + DP_4]\,\delta V_i + [CV_r + DA_r + E]\,\delta t = [0] \tag{3}$$

Solving for δV_i and δt

$$\begin{bmatrix} \delta V_i \\ \delta t \end{bmatrix} = N^{-1} M\,\delta R_i \tag{4}$$

where N and M are 4×4 and 4×3 matrices equal to

$$\begin{aligned} N &= [B + CP_2 + DP_4 \quad CV_r + DA_r + E] \\ M &= [A + CP_1 + DP_3] \end{aligned} \tag{5}$$

Partitioning $N^{-1}M$ into a 3×3 matrix F and a 1×3 row vector G results in the fundamental guidance matrix and the corresponding time guidance vector.

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* The use of δt as a variable and the corresponding use of the time transition matrix is arbitrary. For example, range angle and the corresponding angle transition matrix could have been used.

$$\delta V_i = F\,\delta R_i$$
$$\delta t = G\,\delta R_i \tag{6}$$

where $N^{-1}M = \begin{bmatrix} F \\ G \end{bmatrix}$

III. Three Independent Constraints and a Function to be Minimized

In some cases it is desired to minimize a quantity Q while satisfying three independent scalar constraints. The quantity Q can be generally expressed

$$Q = f(\delta V_i,\ \delta R,\ \delta V,\ \delta t) \tag{7}$$

The perturbations in δR and δV can be expressed as functions of δV_i and δt using Eq. (2).

The three independent constraints can be expressed as in Eq. (1) and combined with Eq. (2) to obtain (see Eq. (3))

$$\mathbf{H}\boldsymbol{\delta}\mathbf{R}_i + K\delta V_i + L\delta t = [0]$$
$$\delta V_i = -K^{-1}H\delta R_i - K^{-1}L\delta t \tag{8}$$

where H and K are 3×3 matrices and L is a column vector. Upon substituting Eq. (8) into Eq. (7), Q is obtained as a function of δt. The derivative of Q with respect to δt defines the value of δt which minimizes Q. If this derivative fails to yield a value for δt, no linear solution exists.

Frequently, the magnitude of a vector is to be minimized and the solution is known to exist. In this case it is more convenient to differentiate the quantity before carrying out the above mentioned substitutions. Differentiating Q with respect to δt results in

$$\boldsymbol{\omega} = S\,\frac{d(\delta V_i)}{d(\delta t)} + T\,\frac{\mathbf{d}(\boldsymbol{\delta}\mathbf{R})}{d(\delta t)} + U\,\frac{d(\delta V)}{d(\delta t)} \tag{9}$$

where S, T and U are row vectors and $\boldsymbol{\omega}$ is a scalar.

Eq. (8) can be differentiated to obtain

$$\frac{d(\delta V_i)}{d(\delta t)} = -K^{-1}L \tag{10}$$

Using Eq. (2)

$$\frac{d(\delta R)}{d(\delta t)} = P_2 \frac{d(\delta V_i)}{d(\delta t)} + V_r$$

$$\frac{d(\delta V)}{d(\delta t)} = P_4 \frac{d(\delta V_i)}{d(\delta t)} + A_r \tag{11}$$

Combining Eqs. (9), (10) and (11)

$$\omega = - S K^{-1} L + T(-P_2 K^{-1} L + V_r) + U(-P_4 K^{-1} L + A_r) \tag{12}$$

The above equation can be directly solved for δt which can be then inserted into Eq. (9).

IV. Examples of Constraints

The following examples of independent constraints lead to the coefficients in Eq. (1) by direct comparison.

a) Terminal position velocity (representing three independent constraints).

$$\delta R = 0 \tag{13}$$

b) Terminal velocity vector (representing three independent constraints).

$$\delta V = 0 \tag{14}$$

c) Terminal position vector that lies in a specified plane.

To obtain a position vector that lies in a specified plane, the component of the final position vector perpendicular to the plane must equal zero. If I is the unit normal to the plane

$$d(I^T R) = 0, \quad I^T \delta R = 0 \tag{15}$$

Note that I need not be a unit vector

d) Terminal velocity vector that lies in a specified plane. If I is the unit normal to the plane

$$d(I^T V) = 0, \quad I^T \delta V = 0 \tag{16}$$

e) Time of flight.

$$\delta t = 0 \tag{17}$$

f) Terminal flight path angle.

$$d(\gamma) = d(\cos^{-1}(r^{-1}v^{-1}R^T V)),$$

$$(V - r^{-2}R^T VR)^T \delta R + (R - v^{-2}R^T VV)^T \delta V = 0 \tag{18}$$

g) Flight through constant radial distance.

$$d(r - r_i) = 0,$$

$$-r^{-1}R^T \delta R - r_i^{-1}R_i^T \delta R_i = 0 \tag{19}$$

h) Period

Maintaining the period of the orbit requires that the differential of the semimajor axis equals zero.

$$d(a) = d(r_i \mu(2\mu - v_i^2 r_i)^{-1}) = 0,$$

$$\mu R_i^T \delta R_i + r_i^3 V_i^T \delta V_i = 0 \tag{20}$$

V. Example of a Reentry Problem

The following constraints are typical of those which might be imposed in deriving guidance laws for reentry problems: maintain the reentry angle, altitude and velocity while minimizing the distance from the nominal reentry point. First, the guidance constraint matrices (redefining them as 3×3 matrices) in Eq. (1) are derived using the independent constraints.

The reentry angle constraint leads to Eq. (18).

$$E_1 = 0,\ A_{1j} = B_{1j} = [0\ 0\ 0] \equiv [\phi]$$

$$C_{1j} = (V - r^{-2}R^T VR)^T$$

$$D_{1j} = (R - v^{-2}R^T VV)^T$$

$$\text{where } j = 1, 2, 3 \tag{21}$$

The reentry altitude constraint utilizes a vector normal to the horizontal plane equal to R in Eq. (15). The result is

$$E_2 = 0, \; A_{2j} = B_{2j} = D_{2j} = [\phi]$$
$$C_{2j} = R^T \tag{22}$$

The reentry velocity constraint utilizes a unit vector colinear with the velocity vector in Eq. (16).

$$E_3 = 0, \; A_{3j} = B_{3j} = C_{3j} = [\phi]$$
$$D_{3j} = V^T \tag{23}$$

Using Eqs. (3) and (8)

$$K = C\,P_2 + D\,P_4$$

$$L = C\,V_n + D\,A_n$$

The distance from the normal reentry point is represented by δr. Differentiating with respect to δt, setting equal to zero and comparing with Eq. (7) gives

$$\omega = 0, \quad S = U = [\phi]$$
$$T = \delta R^T \tag{24}$$

From Eq. (12)

$$0 = \delta R^T(-P_2K^{-1}L + V_n) \tag{25}$$

Using Eq. (2)

$$\delta t = \frac{-(P_1\delta R_i + P_2\delta V_i)^T(-P_2K^{-1}L + V_n)}{V_n^T(-P_2K^{-1}L + V_n)} \tag{26}$$

Inserting into Eq. (9) results in δV_i and the velocity correction.

VI. Example of an Entry into Circular Orbit Problem

The following constraints might be imposed for attaining a circular orbit about the primary body: maintain the nominal flight time, the impulse to enter the circular orbit and the plane normal to the orbit containing the terminal position vector while minimizing the velocity correction.

To maintain the impulse to enter the circular orbit the differential of the difference in the circular velocity V_c and the final velocity must equal zero.

$$\begin{aligned} d((V_c - V)^T(V_c - V))^{1/2} &= 0 \\ (V_c - V)^T(\delta V_c - \delta V) &= 0 \end{aligned} \tag{27}$$

where

$$\delta V_c = -.5\mu^{1/2} r^{-9/2} v^{-1} [R \times (V \times R)] R^T \delta R$$

Comparing with Eq. (1)

$$E_1 = 0, \; A_{1j} = B_{1j} = [\phi]$$

$$C_{1j} = -.5\mu^{1/2} r^{-9/2} v^{-1} (V_c - V)^T [R \times (V \times R)] R^T \tag{28}$$

$$D_{1j} = -(V_c - V)^T$$

The constraint of maintaining the terminal plane utilizes a normal to that plane given by $R \times (V \times R)$. Noting Eq. (15)

$$\begin{aligned} E_2 &= 0, \; A_{2j} = B_{2j} = D_{2j} = [\phi] \\ C_{2j} &= [R \times (V \times R)]^T \end{aligned} \tag{29}$$

Maintaining the time of flight requires that δt equal zero. Hence,

$$\begin{aligned} A_{3j} &= B_{3j} = C_{3j} = D_{3j} = [\phi] \\ E_3 &= 1 \end{aligned} \tag{30}$$

Using Eq. (3) and (8)

$$H = C P_1 + D P_3$$

$$K = C P_2 + D P_4 \tag{31}$$

$$L = C V_n + D A_n + E$$

Minimizing the velocity correction with respect to δt results in

$$(\delta V_i - \delta V_{initial})^T \frac{d(\delta V_i)}{d(\delta t)} = 0 \tag{32}$$

Comparing with Eq. (9)

$$\omega = 0, \quad T = U = [\phi]$$

$$S = (\delta V_i - \delta V_{initial})^T \tag{33}$$

From Eq. (12)

$$0 = - S K^{-1} L \tag{34}$$

Using Eqs. (8), (31) and (33)

$$\delta t = \frac{-(K^{-1}L)^T (K^{-1}H)\, \delta R_i - (K^{-1}L)^T\, \delta V_{initial}}{(K^{-1}L)^T K^{-1} L} \tag{35}$$

Variable arrival time guidance utilizes the terminal position vector constraint given in Eq. (13) along with the constraint of minimizing the velocity correction. Hence C is a unity matrix, H, K, and L equal P_1, P_2 and V_t (assuming a target trajectory), respectively, and δt is obtained from Eq. (35).

Note that coplanar problems leave only three constraints to be imposed.