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Space Guidance Analysis Memo #2-69

TO: Distribution
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SUBJECT: INCLUSION OF BIASES IN STATISTICAL MIDCOURSE NAVIGATION

Because sextant and horizon biases may be important in midcourse navigation, and since non-linear problems do not seem to be of major consequence in midcourse, the following derivation has been carried out. With the following equations it should be possible to avoid doing most Monte Carlo (MC) simulations, although as a final check, MC studies are always desirable.

We know the error in the state vector after the K^{th} measurement is

$$\underline{e}_K = \underline{e}'_K - \underline{w}_K (\underline{b}_K \underline{e}'_K - (\alpha_K + \beta_K)) \tag{1}$$

where ' indicates before the measurement and α_K is the random (from measurement to measurement) measurement error; i.e., $\overline{\alpha_J \alpha_K} = 0$, and β_K is the correlated (colored) part of the measurement noise: $\overline{\beta_J \beta_K}$ can be non-zero. The vectors $\underline{w}_K$ and $\underline{b}_K$ are the weighting and geometry vectors.

Taking the transpose of $\underline{e}_K$ and forming the covariance matrix, we have

$$E_K = \overline{\underline{e}_K \underline{e}_K^T} = (I - \underline{w}_K \underline{b}_K^T) E'_K (I - \underline{w}_K \underline{b}_K^T)^T + \underline{w}_K \underline{w}_K^T (\overline{\alpha_K^2} + \overline{\beta_K^2})$$

$$+ (I - \underline{w}_K \underline{b}_K^T) \overline{\underline{e}'_K \beta_K} \underline{w}_K^T + \underline{w}_K \overline{\beta_K \underline{e}'^T_K} (I - \underline{w}_K \underline{b}_K^T)^T \tag{2}$$

Since we desire to know what the true uncertainties are (neglecting non-linear effects), $\underline{w}_K$, the weighting vector which must be used in Eq. 2 is

$$\underline{w}_K = \frac{E'_{OB,K}\,\underline{b}_K}{\underline{b}^T_K\,E'_{OB,K}\,\underline{b}_K + \overline{\alpha^2}_{OB,K}}$$

where E_{OB} is the on-board covariance matrix. This $\underline{w}_K$ must be used in the calculation of E (the true covariance). The on-board covariance matrix nominally commences with the approximation

$$E_{OB} = \begin{pmatrix} P & & & & & O \\ & P & & & & \\ & & P & & & \\ & & & V & & \\ O & & & & V & \\ & & & & & V \end{pmatrix}$$

The true matrix, E, will commence with the true injection uncertainties. Also, the true $\overline{\alpha^2}$ may be different than the on-board $\overline{\alpha^2}_{OB}$. Hence, everything is known and can be calculated in Eq. 2 except for the last two terms which involve the unknown vector, $\overline{\underline{e}'_K\,\beta_K}$. Since β_K is the colored part of the measurement noise, it can possibly be correlated with all of the previous β's that are implicitly included in $\underline{e}'_K$.

The equation for the error before the K^{th} measurement is from Eq. 1,

$$\underline{e}'_K = \Phi_{K,K-1}\left[(I - \underline{w}_{K-1}\,\underline{b}^T_{K-1})\,\underline{e}'_{K-1} + \underline{w}_{K-1}\,(\alpha_{K-1} + \beta_{K-1})\right] \quad (3)$$

which is just $\underline{e}_{K-1}$ propagated up to the time, K, with the transition matrix, Φ (assuming no driving noise).

If we let $\underline{e}_0$ be the injection error, then just before the first measurement

$$\underline{e}'_1 = \Phi_{1,0} \; \underline{e}_0 \tag{4}$$

After the first measurement the error is:

$$\underline{e}_1 = \underline{e}'_1 + \underline{w}_1 (\alpha_1 + \beta_1 - \underline{b}_1^T \underline{e}'_1)$$

and using Eq. (4);

$$\underline{e}_1 = (I - \underline{w}_1 \underline{b}^T_1) \Phi_{1,0} \underline{e}_0 + \underline{w}_1 (\alpha_1 + \beta_1)$$

Extrapolating to the second measurement gives

$$\underline{e}'_2 = \Phi_{2,1} \underline{e}_1 = \Phi_{2,1} \left[(I - \underline{w}_1 \underline{b}^T_1) \Phi_{1,0} \underline{e}_0 + \underline{w}_1 (\alpha_1 + \beta_1) \right]$$

The error before the next measurement is

$$\underline{e}'_3 = \Phi_{3,2} \Big[(I - \underline{w}_2 \underline{h}^T_2) \Phi_{2,1} \big((I - \underline{w}_1 \underline{b}_1) \Phi_{1,0} \underline{e}_0 + \underline{w}_1 (\alpha_1 + \beta_1) \big) + \underline{w}_2 (\alpha_2 + \beta_2) \Big] \; ;$$

etc.

From these above equations and Eq. 3, there follows

$$\underline{e}'_K = \Phi_{K,K-1}\left\{(I - \underline{w}_{K-1} b^T_{K-1})\,\Phi_{K-1,K-2}\left[(I - \underline{w}_{K-2}\underline{b}^T_{K-2}) \ldots\right.\right.$$

$$\Phi_{3,2}\left((I - \underline{w}_2\,\underline{b}^T_2)\,\Phi_{2,1}\left\{(I - \underline{w}_1\underline{b}^T_1)\,\Phi_{1,0}\,\underline{e}_0 + \underline{w}_1(\alpha_1 + \beta_1)\right\}\right.$$

$$\left.\left.\left. + \underline{w}_2(\alpha_2 + \beta_2)\right) + \ldots. + \underline{w}_{K-2}(\alpha_{K-2} + \beta_{K-2})\right] + \underline{w}_{K-1}(\alpha_{K-1} + \beta_{K-1})\right\}$$

or

$$\underline{e}'_K = \underbrace{\prod_{\substack{I=1\\K>1}}^{I=K-1} \Phi_{I+1,I}(I - \underline{w}_I\underline{b}^T_I)\,\Phi_{1,0}\,\underline{e}_0}_{} + \underbrace{\Phi_{1,0}\,\underline{e}_0}_{K=1\ \text{only}} + \underbrace{\Phi_{K,K-1}\,\underline{w}_{K-1}(\alpha_{K-1} + \beta_{K-1})}_{K>1}$$

$$+ \sum_{\substack{I=1\\K>2}}^{I=K-2}\left[\prod_{J=I+1}^{J=K-1}\Phi_{J+1,J}(I - \underline{w}_J\underline{b}^T_J)\right]\Phi_{I+1,I}\,\underline{w}_I(\alpha_I + \beta_I)$$

where in the products, a factor with I or J = M-1 is to the right of the factor with I or J = M.

Post multiplying by β_K and taking the expectation gives

$$\overline{\underline{e}'_K\beta_K} = C_1\,\overline{\underline{e}_0\beta_K} + \Phi_{1,0}\,\overline{\underline{e}_0\beta_K} + \Phi_{K,K-1}\,\underline{w}_{K-1}\left[\overline{\alpha_{K-1}\beta_K} + \overline{\beta_{K-1}\beta_K}\right] \qquad (5)$$

$$+ C_2\left[\overline{\alpha_I\beta_K} + \overline{\beta_I\beta_K}\right].$$

The first two terms contain $\overline{\underline{e}_0\beta_K}$ and are zero since $\underline{e}_0$ is assumed to be independent of all measurement errors. Also, since α_I is assumed to be completely random, $\overline{\alpha_I\beta_K} = 0$ for all I and K. For our present study, we shall make another simplification and assume $\overline{\beta_I \beta_K}$ is not dependent on I (on time); therefore, $\overline{\beta_I \beta_K} = \overline{\beta^2} = \sigma_\beta^2$ where σ_β is the standard deviation. Other models for the time dependence of β are possible. A common one is of the form e^{-at} where a is a positive constant.

So, Eq. 5 becomes

$$\overline{\underline{e}'_K \beta_K} = \Big\{ \underset{K > 1}{\Phi_{K,K-1} \underline{w}_{K-1}}$$

$$+ \sum_{\substack{I=1 \\ K>2}}^{I=K-2} \Big[\prod_{J=I+1}^{J=K-1} \Phi_{J+1,J} (I - \underline{w}_J \underline{b}_J) \Big] \Phi_{I+1,I} \underline{w}_I \Big\} \sigma_\beta^2 \qquad (6)$$

If there are both instrument and horizon biases, the variance can be written as

$$\sigma_\beta^2 = \sigma_{I\beta}^2 + \sigma_{H\beta}^2$$

where $\sigma_{I\beta}$ is the instrument bias standard deviation and $\sigma_{H\beta}$ is the horizon bias standard deviation of the planet. We have assumed no time dependence of either bias and the independence of each from the other.

At MCC times, if the direction and magnitude of the acceleration measurement uncertainties due to the burns are known, then these can be added to the true covariance matrix (E) at each MCC.

A simple recursion equation can be obtained from Eq. 6 if it is written in a compiler language instead of as an algebraic expression. In MAC, at the K th measurement, the equation for $\overline{\underline{e}'_K \beta_K}$ can be written as

$$\overline{A} = \overset{*}{PHI} \left((\overset{*}{I} - \overline{W}_P \overline{B}_P) \overline{A} + \overline{W}_P \, BIAS^2 \right) \quad (7)$$

where $\overline{A}$ on the left side is $\overline{\underline{e}'_K \beta_K}$, and $\overline{A}$ on the right side is $\overline{\underline{e}'_{K-1} \beta_{K-1}}$. $\overset{*}{PHI}$ is $\Phi_{K,K-1}$, and $\overline{W}_P$ and $\overline{B}_P$ are the weighting and geometry vectors from the previous (K-1) measurement. BIAS is the standard deviation of the bias. Equation 7 is valid at all measurements except the first one. For the first measurement, A = 0.