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Space Guidance Analysis Memo #2-65

TO: SGA Distribution
FROM: William Marscher
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SUBJECT: A Modified Encke

Introduction

The purpose of this memo is to outline a modified Encke method for generating precision trajectories which is computationally faster than the conventional Encke. The modification is a change of the independent variable from time to the universal conic variable "x". The integration of the resulting equations does not require the time consuming iterative solution of Kepler's equation.

The idea of changing the independent variable is not new*, but the particular formulation of the differential equations presented herein is.

This memo will first contain an outline of the conventional Encke for reference; then the change of variables will be made followed by a discussion of the resulting new equations along with a numerical comparison between the Encke and the modified Encke.

The Conventional Encke**

The six equations to be integrated in the conventional Encke are:

$$\frac{d\,\delta\bar{r}}{dt} = \delta\,\bar{v} \tag{1}$$

$$\frac{d\,\delta\bar{v}}{dt} = \frac{\mu}{r_c^{\,3}}\left[f(q)\,\bar{r} + \delta\bar{r}\right] + \bar{a}_d \tag{2}$$

*See References 1 and 2

**See Reference 1, Chapter 6 for derivations where:

μ	=	Mu of the primary body
t	=	Time
$\overline{r}_c$	=	Osculating conic position
$\overline{v}_c$	=	Osculating conic velocity
$\overline{r}$	=	True position
$\overline{v}$	=	True velocity
$\delta\overline{r}$	=	Encke position deviation, $\overline{r} - \overline{r}_c$
$\delta\overline{v}$	=	Encke velocity deviation, $\overline{v} - \overline{v}_c$
$\overline{a}_d$	=	Acceleration due to disturbing bodies usually a function of r and t

$$f(q) = \frac{q(3 + 3q + q^2)}{1 + (1 + q)^{3/2}}$$

$$q = \frac{(\delta\overline{r} - 2\overline{r}) \cdot \delta\overline{r}}{\overline{r} \cdot \overline{r}}$$

The computation of $\overline{r}_c$ and $\overline{v}_c$ is made by first iteratively solving Kepler's equation below, for "x"

$$\sqrt{\mu}\, t_r = \frac{\overline{r}_0 \cdot \overline{v}_0}{\sqrt{\mu}} x^2 C(\alpha x^2) + (1 - r_0 \alpha) x^3 S(\alpha x^2) + r_0 x \qquad (3)$$

where

t_r	=	Time since rectification
$\overline{r}_0$	=	Position at rectification
$\overline{v}_0$	=	Velocity at rectification
α	=	$1/a$, a = semimajor axis based on $\overline{r}_0$ and $\overline{v}_0$

And then explicitly computing $\overline{r}_c$ and $\overline{v}_c$ where

$$\bar{r}_c = \left[1 - \frac{x^2}{r_0} C(\alpha x^2)\right] \bar{r}_0 + \left[t_r - \frac{x^3}{\sqrt{\mu}} S(\alpha x^2)\right] \bar{v}_0 \qquad (4)$$

$$\bar{v}_c = \frac{\sqrt{\mu}}{r_c\, r_0} \left[\alpha x^3 S(\alpha x^2) - x\right] \bar{r}_0 + \left[1 - \frac{x^2}{r_c} C(\alpha x^2)\right] \bar{v}_0 \qquad (5)$$

<u>Change of Independent Variable</u>

The independent variable will now be changed from "t" to "x". This is done to eliminate the iterative solution of Eq. (3).

The change is easily made by recognizing that

$$\frac{d\,\delta\bar{r}}{dx} = \frac{d\,\delta\bar{r}}{dt} \frac{dt}{dx} \qquad (6)$$

$$\frac{d\,\delta\bar{v}}{dx} = \frac{d\,\delta\bar{v}}{dt} \frac{dt}{dx} \qquad (7)$$

$$\frac{dt}{dx} = \frac{r_c}{\sqrt{\mu}} \qquad (8)$$

Thus x becomes the independent variable by simply multiplying (1) and (2) by (8). The remainder of the computation remains the same with the exception that time can now be computed explicitly using Eq. (3). Should it be desired to terminate the integration at a desired time, it is a simple matter to test the computed time and, if it exceeds the desired time, to solve Eq. (3), as in the conventional Encke, iteratively for the "x" which will produce the desired time.

In the conventional Encke, the allowable integration time step for a constant truncation error was found to be*:

$$\Delta t = \frac{K}{\sqrt{\mu}} r_b^{3/2} \qquad (9)$$

where:

Δt	=	Max allowable integration step
K	=	Emperical constant
r_b	=	Distance from primary body

*See Appendix

K is established by a doubling and halving integration. The corresponding "x" step can be computed to be (substituting Eq. (8) into Eq. (9)):

$$\Delta x = K\, r^{1/2} \qquad (10)$$

Thus, the Δx step, which varies as $r^{1/2}$, produces the same time step as the conventional Encke produces. Also, it is interesting to note that the Δx step is independent of mu, which the time step was not.

Numerical Comparison of the Modified Encke and the Conventional Encke

The modified Encke was run in the doubling and halving mode with the same initial conditions and for the same time of flight that produced the results outlined in SGA Memo #60-64 for the conventional Encke. The trajectory travels from the earth to the moon's sphere of influence in approximately 58 hours.

The following is a comparison of the results:

a) Number of Time Steps:

Conventional Encke ~ 40

Modified Encke ~ 40

b) Final Conditions (Miles and Hours):

Conventional Encke Position	204024.7817	85024.97463	37052.37272
Modified Encke Position	204025.0081	85025.00914	37052.37926
Difference Encke Position	-0.2264	-.02451	-.00654
Conventional Encke Velocity	1861.927370	1118.695832	565.9181469
Modified Encke Velocity	1861.932431	1118.697141	565.9186049
Difference Encke Velocity	-.005061	-.001309	-.0004580

c) Time Step History

The attached curve presents the time step history as a function of distance from the earth for both the conventional and modified Encke, along with the Δx step history. Note that Δt and Δx varies approximately as $r^{3/2}$ and $r^{1/2}$ respectively. Which of the two methods is more accurate is a moot point.

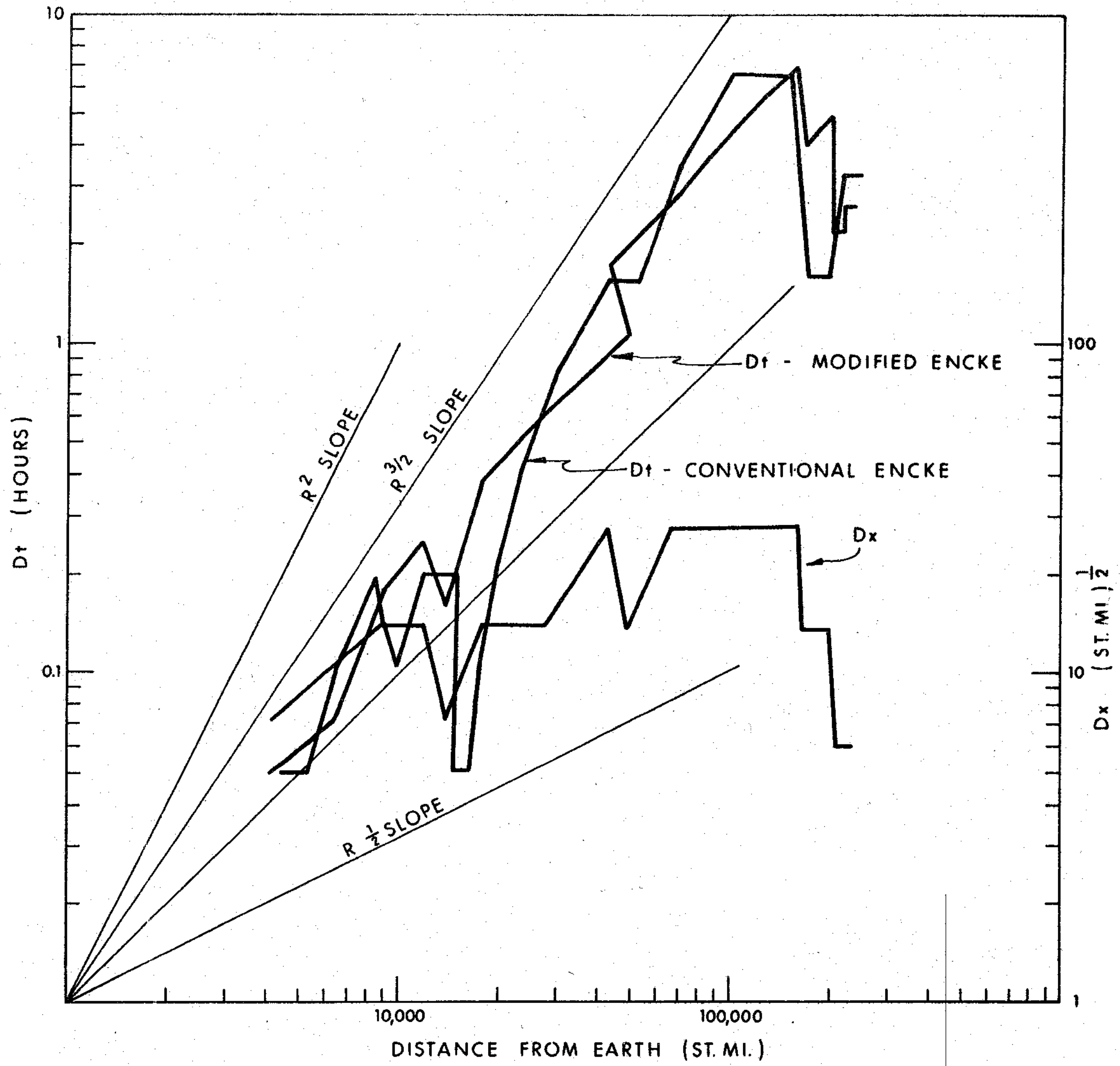

Fig. 1 Comparison of the modified encke and the conventional encke using a translunar trajectory.

References

1. Battin, R. H., Astronautical Guidance, McGraw-Hill Book Company, Inc., New York, New York, 1964.
2. Ralston and Wilf, Editors, Mathematical Methods for Digital Computers, John Wiley and Son, Inc., New York, New York, 1960.
3. Moore, W. F., An Encke Method Adapted to Mission Analysis, Sept. 2, 1964, Unpublished.

Appendix

One can, without too much trouble, convince oneself that Eq. (9) is proper.

The truncation error resulting from the numerical integration of a first order differential equation can be expressed as follows:*

$$e \approx k \, \Delta t^{p+1} \, f^{p+1} \tag{1a}$$

where

p	=	the order of the integration
e	=	truncation error
k	=	error constant, dependent on integration method
Δt	=	independent variable
f^{p+1}	=	(p + 1) th derivative of the equation be integrated

One can let p = 4 and select the following equations, which are the very familiar two dimensional equations of motion of a conic, to illustrate the point:

$$\frac{d r_x}{dt} = \sqrt{\frac{\mu}{p}} \, \frac{e}{2} \sin(2f) - \frac{\sqrt{\mu p}}{r} \sin(f) \tag{2a}$$

$$\frac{d r_y}{dt} = \sqrt{\frac{\mu}{p}} \, e \sin^2(f) + \frac{\sqrt{\mu p}}{r} \cos(f) \tag{3a}$$

$$\frac{d v_x}{dt} = - \frac{\mu \cos(f)}{r^2} \tag{4a}$$

$$\frac{d v_y}{dt} = - \frac{\mu \sin(f)}{r^2} \tag{4a}$$

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*See Reference 2

If Eq. (2a) were differentiated four more times and substituted into Eq. (la), one would find that

$$\Delta t \approx \left(\frac{e}{k}\right)^{1/5} \frac{f(r,\ p,\ a)}{\sqrt{\mu}} \qquad (6a)$$

where p and a are conic parameters and f (r, p, a) includes $r^{9/5}$ to $r^{6/5}$.

Since for a given trajectory p and a are constants, one might conjecture that, with constant truncation error, Eq. (6a) can be approximated by:

$$\Delta t \approx \frac{k\ r^n}{\sqrt{\mu}} \qquad (7a)$$

where

k = an emperical constant

n = an emperical power of r

Experience has shown the conjecture to be true and that $n \approx 3/2$.

The same conclusions would also be reached for Eqs.(3a), (4a) and (5a).

This analysis, of course, applies more directly to the Cowell method than to the Encke; however, the right hand side of the Encke differential equations also are essentially functions of r^{-2} and r^{-1} and mu couples into the equations in the same fashion. Without carrying out a rather complex analysis, one can again conjecture that Eq. (7a) might apply to the Encke also. Experience again has borne out the conjecture.

Experience has also shown that the effect of changing p and a, for practical earth and moon centered trajectories is negligible and that, in truth, the time step can be ratioed as $mu^{1/2}$.