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Space Guidance Analysis Memo #1-64

TO: SGA Distribution

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SUBJECT: Effect of Correlated Measurement Errors on Midcourse Performance

This memo presents the effect of correlated measurement errors on midcourse performance. In particular this study investigates the effect of correlated measurement errors on earth to moon flight using the basic treatment for statistical simulation of space flight given by Dr. Battin.

Part I presents the necessary theory. Part II investigates earth moon flights where we assume zero correlation between measurement errors which is not correct. The results will then show how serious the assumption of uncorrelated measurement errors is on final uncertainties, deviations and total velocity required.

Part I

First we will present the equations which the spacecraft computer will use with the assumed correlation factor. Then we will present the equations resulting from using the true correlation factor.

The deviation vector is seven dimensional

$$\delta \underline{x}_n = \begin{bmatrix} \delta \underline{r}_n \\ \delta \underline{v}_n \\ \alpha_n \end{bmatrix}$$

and the error vector is also seven dimensional

$$\underline{e}_n = \begin{bmatrix} \underline{\epsilon}_n \\ \underline{\delta}_n \\ \beta_n \end{bmatrix}$$

where

$$\hat{\alpha}_n = \alpha_n + \beta_n$$

β_n can be considered the error in the estimation of the measurement error and α_n is the measurement error. Let $\underline{\xi}_n$ be a seven dimensional vector with the first six terms zero and the last being the random measurement error.

(A) Calculations Using Assumed Correlation Factor

(1) Extrapolating E, X matrices

The transition matrix for extrapolating the assumed seven dimensional covariance matrices of deviation and error is

$$P^*_{n,\,n-1} = \begin{bmatrix} \Phi_{n,\,n-1} & 0 \\ 0 & COR \end{bmatrix} \quad (1)$$

where COR = assumed correlation factor = 0 and $\Phi_{n,\,n-1}$ = transition matrix of trajectory deviation equations. The two six dimensional zero vectors imply that the measurement errors and trajectory deviations are uncorrelated. The extrapolated correlation matrix of errors is computed from

$$E_n' = P^*_{n,\,n-1} E_{n-1} P^{*T}_{n,\,n-1} + Q$$

where

$$Q = \begin{bmatrix} \begin{matrix} 6 \times 6 \\ \text{zero} \\ \text{matrix} \end{matrix} & 0 \\ 0 & \overline{\zeta_n^2} \end{bmatrix}$$

and $\overline{\zeta_n^2}$ is the variance of the individual sextant measurement considered as a function of time. For studies in this report $\overline{\zeta_n^2}$ = constant. This is an approximation since $\overline{\zeta_n^2}$ depends on distance of spacecraft from the near body. A more accurate model would be

$$\overline{\zeta_n^2} = VARA + VARD/R^2$$

where VARA = constant variance and VARD could be the variance of the planet horizon and R is the distance of the spacecraft from the planet.

The extrapolated correlation matrix of deviations is computed by

$$X_n' = P^*_{n,\,n-1}\, X_n\, P^{*\,T}_{n,\,n-1}$$

(2) Velocity Correction

At a velocity correction the E matrix is updated by

$$E_n = E_n' + L\, N_n\, L^T$$

where L is a 7 × 3 matrix

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $N_n = \overline{\eta_n \eta_n^T}$ (3 × 3) matrix where $\underline{\eta}_n$ is the error in application of a velocity correction at time t_n. For updating the covariance matrix of deviations we have

$$X_n = (I + L B_n)(X_n' - E_n')(I + L B_n)^T + E_n' + L N_n L^T$$

This is the same equation as 9.4-16 in Dr. Battin's class notes, but L changes as above and B_n is now a (3 × 7) matrix

$$B_n = \begin{bmatrix} C_n^* & -I & 0 \end{bmatrix}$$

where C_n^* is a function of the reference trajectory (see Dr. Battin's class notes).

(3) Measurement

At a measurement the E, X matrices are updated as follows

$$E_n = E_n' - a_n^{-1} E_n' \underline{b}_n \underline{b}_n^T E_n'$$

where

$$a_n = \underline{b}_n^T E_n' \underline{b}_n$$

and

$$\underline{b}_n = \begin{bmatrix} \underline{h} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $X_n = X_n'$ at a measurement, $\underline{h}$ is a function of the measurement made.

(B) Calculations Using True Correlation Factor

In part A, the matrix E is not the true covariance matrix of the errors, but the ship's computer would erroneously call it that since it merely follows the formula for optimum smoothing. Part B determines the true E, X matrices and how they are updated for measurements and velocity corrections.

(1) Measurement

The real error in the estimate is

$$\underline{e}_n^{**} = \delta \underline{\hat{x}}_n^* - \delta \underline{x}_n \quad (2)$$

where $\delta \underline{x}_n$ = true deviation vector and $\delta \underline{\hat{x}}_n^*$ = estimate of deviation vector that the computer calculates. Now the extrapolated deviation vector that the computer calculates is

$$\delta \underline{\hat{x}}_n^{*'} = P_{n,\,n-1}^* \, \delta \hat{x}_{n-1}^* \quad (3)$$

where $P_{n,\,n-1}^*$ is given by Eq. (1).

$$\delta \underline{\hat{x}}_n^* = \delta \underline{\hat{x}}_n^{*'} + \underline{w}_n^* (\delta \tilde{A}_n - \delta \hat{A}_n^{*'}) \quad (4)$$

where

$$\delta \tilde{A}_n = \underline{b}_n^T \, \delta \underline{x}_n \quad (5)$$

$$\delta \hat{A}_n^{*'} = \underline{b}_n^T \, \delta \underline{\hat{x}}_n^{*'} = \underline{b}_n^T \, P_{n,\,n-1}^* \, \delta \hat{x}_{n-1}^* \quad (6)$$

Substituting 5, 6 in 4

$$\delta \underline{\hat{x}}_n^* = \delta \underline{\hat{x}}_n^{*'} + \underline{w}_n^* \left(\underline{b}_n^T \, \delta \underline{x}_n - \underline{b}_n^T \, P_{n,\,n-1}^* \, \delta \underline{\hat{x}}_{n-1}^* \right) \quad (7)$$

Substituting 3 in 7

$$\delta \underline{\hat{x}}_n^* = P_{n,n-1}^* \, \delta \underline{\hat{x}}_{n-1}^{*\prime} + \underline{w}_n^* \left(\underline{b}_n^T \, \delta \underline{x}_n - \underline{b}_n^T \, P_{n,n-1}^* \, \delta \underline{\hat{x}}_{n-1} \right) \tag{8}$$

Substituting 8 in 2

$$\underline{e}_n^{**} = P_{n,n-1}^* \, \delta \underline{\hat{x}}_{n-1}^{*\prime} + \underline{w}_n^* \left(\underline{b}_n^T \, \delta \underline{x}_n - \underline{b}_n^T \, P_{n,n-1}^* \, \delta \underline{\hat{x}}_{n-1}^* \right) - \delta \underline{x}_n \tag{9}$$

Now the true deviation vector must be

$$\delta \underline{x}_n = P_{n,n-1} \, \delta \underline{x}_{n-1} + \underline{\xi}_n \tag{10}$$

to satisfy our assumed sextant model for the correlated measurement errors.

$$\alpha_n = \alpha_{n-1} \, \mathrm{CORT} + \xi_n$$

More will be said about the sextant model in Part II

$$P_{n,n-1} \quad \begin{bmatrix} \Phi_{n,n-1} & 0 \\ 0 & \mathrm{CORT} \end{bmatrix}$$

where CORT = true correlation factor.

Substituting 10 in 9

$$\underline{e}_n^{**} = \left(I - \underline{w}_n^* \, \underline{b}_n^T \right) \left[P_{n,n-1}^* \, \delta \underline{\hat{x}}_{n-1}^* - \left(P_{n,n-1} \, \delta \underline{x}_{n-1} + \underline{\xi}_n \right) \right] \tag{11}$$

The true error vector at time t_{n-1} is

$$\underline{e}_{n-1}^{**} = \delta \underline{\hat{x}}_{n-1}^* - \delta \underline{x}_{n-1} \tag{12}$$

Substituting (12) in (11)

$$\underline{e}_n^{**} = \left(I - \underline{w}_n^* \underline{b}_n^T\right) P^*_{n,n-1} \underline{e}_{n-1}^{**} + \left(I - \underline{w}_n^* \underline{b}_n^T\right) \left(P^*_{n,n-1} - P_{n,n-1}\right) \delta \underline{x}_{n-1}$$

$$- \left(I - \underline{w}_n^* \underline{b}_n^T\right) \underline{\xi}_n \qquad (13)$$

Equation (13) gives the true error in the estimate at time t_n in terms of a measurement at time t_n, true error in the estimate at time t_{n-1} and the true deviation at time t_{n-1}. All these quantities are known, so $\underline{e}_n^{**}$ the true error in the estimate at time t_n can be determined. Now

$$\underline{w}_n^* = a_n^{*-1} E_n^{*'} \underline{b}_n$$

where

$$a_n^{*-1} = \underline{b}_n^T E_n^{*'} \underline{b}_n$$

For compactness, let

$$M = \underline{w}_n^* \underline{b}_n^T$$

$$D = I - M$$

$$B = D(P^*_{n,n-1} - P_{n,n-1})$$

$$A = D P^*_{n,n-1}$$

Then (13) becomes

$$\underline{e}_n^{**} = A \underline{e}_{n-1}^{**} + B \delta \underline{x}_{n-1} - D \underline{\xi}_n$$

Let

F = true covariance matrix of the error in the estimate

S = true covariance matrix of the deviation

Consequently using (13)

$$F_n = \overline{\underline{e}_n^{**} \underline{e}_n^{**T}} = A F_{n-1} A^T + B S_{n-1} B^T + D Q D^T$$

$$+ B \overline{\delta \underline{x}_{n-1} \underline{e}_{n-1}^{**T}} A^T + A \overline{\underline{e}_{n-1}^{**} \delta \underline{x}_{n-1}^T} B^T \tag{14}$$

Since

$$\overline{\underline{e}_{n-1}^{**} \underline{\zeta}_n^T} = \overline{\delta \underline{x}_{n-1} \underline{\zeta}_n^T} = 0 \quad (\underline{\zeta}_n \text{ assumed independent})$$

$$S_n = \overline{\delta \underline{x}_n \delta \underline{x}_n^T} = \overline{\left(P_{n,n-1} \delta \underline{x}_{n-1} + \underline{\zeta}_n\right) \left(P_{n,n-1} \delta \underline{x}_{n-1} + \underline{\zeta}_n\right)^T}$$

$$S_n = P_{n,n-1} S_{n-1} P_{n,n-1}^T + Q \tag{15}$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \overline{\zeta_n^2} \end{bmatrix}$$

The problem is to evaluate

$$\overline{\underline{e}_{n-1}^{**} \delta \underline{x}_{n-1}^T}$$

which is the covariance matrix between the true error in the estimate and the true deviation. At launch, we assume

$$\delta \hat{\underline{x}} = 0 = \underline{e}_L = \delta \underline{x}_L$$

$$\underline{e}_L = - \delta \underline{x}_L$$

$$\delta \hat{\underline{x}} = 0 = \underline{e}_L^{**} + \delta \underline{x}_L$$

$$e_L^{**} = - \delta \underline{x}_L$$

Consequently

$$E_L = F_L = X_L = S_L \tag{16}$$

therefore,

$$\overline{\underline{e}_L^{**} \, \delta \underline{x}_L^T} = \overline{\underline{e}_L^{**} (-\underline{e}_L^{**T})} = -F_L \tag{17}$$

Note that 16, 17 are true only at launch. Equations 16, 17 are the initial conditions which enable us to start a calculation. When we make a measurement we must be able to evaluate

$$G = \overline{\underline{e}_L^{**} \, \delta \underline{x}_L^T}$$

and also at a velocity correction which will be given later. Let

$$K = \overline{\delta \underline{x}_L \, \underline{e}_L^{**T}} = G^T$$

$$G_n = \overline{\underline{e}_n^{**} \, \delta \underline{x}_n^T} = A \, \overline{\underline{e}_{n-1}^{**} \, \delta \underline{x}_{n-1}^T} \, P^T + B \, S_{n-1} \, P^T - DQ$$

$$G_n = A G_{n-1} P^T + B S_{n-1} P^T - DQ \tag{18}$$

$$K_n = G_n^T = P^T (KA^T + SB^T) - QD^T \tag{19}$$

Equations 14, 15, 18 and 19 are calculated at every measurement point. The same must be done at a velocity correction point.

(2) Velocity Correction

The F and S matrices are updated at velocity correction as in Section A

$$F_n = F_n' + L N_n L^T$$

$$S_n = (I + L B_n)(S_n' - F_n')(I + L B_n)^T + F_n' + L N_n L^T$$

We must also determine how to update G at a velocity correction. At a velocity correction

$$\delta \underline{x} = \delta \underline{x}' + L B \delta \underline{\hat{x}}_n' - L \underline{\eta}_n$$

$$\overline{\underline{e}^{**} \delta \underline{x}^T} = \overline{\underline{e}^{**}(\delta \underline{x}' + L B \delta \underline{\hat{x}}_n' - L \underline{\eta}_n)^T} = \overline{\underline{e}^{**} \delta \underline{x}'^T}$$

Since $\underline{\eta}_n$ was assumed independent and $\overline{\underline{e}^{**} \delta \underline{\hat{x}}_n} = 0$ (See Section 9.3 of Dr. Battin's class notes). Therefore G (after velocity correction) = G (before velocity correction) and G before a velocity correction is simply updated with the true transition matrix

$$G_n = G_n' = P_{n,n-1} G_{n-1} P_{n,n-1}^T$$

(3) Extrapolating F, S matrices

The F, S matrices are extrapolated as in Section A but using the true transition matrix

$$F_n' = P_{n,n-1} F_{n-1} P_{n,n-1}^T + Q$$

$$S_n' = P_{n,n-1} S_{n-1} P_{n,n-1}^T$$

Part II

Application to Earth Moon Flight

For earth to moon flight, the following assumptions were made:

(1) $X_L = E_L$

(2) The model assumed for correlated measurement errors is

$$\alpha_n = \alpha_{n-1} \exp [-\lambda (t_n - t_{n-1})] + \zeta_n$$

where correlation factor equals

$$\exp [-\lambda (t_n - t_{n-1})]$$

and α_n, ζ_n are independent random variables, λ a positive constant and $\overline{\zeta_n}$ equals zero. The first term on the right can be considered a drift term and the second term is the random error of the actual sextant observation.

(3) $T = t_n - t_{n-1}$ = constant = 1.5 hrs. This number was selected as a first approximation since the flight time is 62.5 hours and 38 measurements were taken.

(4) $\overline{\zeta_n^2} = \overline{\zeta^2}$ = constant = 10 seconds. As stated in Part I this is not exact since horizon uncertainty is a function of distance.

(5) Uncertainty in altitude = .96 miles
Uncertainty in range = 1.43 miles
Uncertainty in track = .37 miles
Uncertainty in altitude = 7.79 mph
Uncertainty in range = 4.43 mph
Uncertainty in track = 5.08 mph

or

$$
E_L = \begin{bmatrix}
1.88 & .34 & .25 & 0 & 0 & 0 & 0 \\
.34 & .87 & .41 & 0 & 0 & 0 & 0 \\
.25 & .41 & .37 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 26.14 & -13.23 & -7.10 & 0 \\
0 & 0 & 0 & -13.23 & 48.42 & 11.56 & 0 \\
0 & 0 & 0 & -7.10 & 11.56 & 31.79 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The zero in the last diagonal term implies that we have perfect knowledge of α_L, the initial sextant bias. Therefore β_L is zero since β is the error in the estimation of measurement error.

Existing programs using dimension six were utilized by adding subroutines whenever the dimension seven was used in the calculations. MAC is now capable of matrix operations up to dimension seven.

The moon earth calculations were made for 38 measurements at fixed times with three velocity corrections at 8, 49 and 61.6 hours. Six cases were investigated with true correlation factor equal to 0, .25, .50, .75, .85, 1.0 and a zero assumed correlation factor. Figures 1 to 5 presents the final uncertainties, deviations and velocity required as a function of the true correlation factor.

The object of this study is to relate this data to actual sextants. This is accomplished by determining the correlation factor as a function of the steady state mean square drift of the sextant. First we obtain the steady state value of $\overline{\alpha_n^2}$. Squaring α_n and averaging gives

$$\overline{\alpha_n^2} = \overline{\alpha_{n-1}^2} \exp(-2\lambda\Delta T) + \overline{\zeta_n^2}$$

$\overline{\zeta_n^2} = \overline{\zeta^2}$ = constant and CORT = $e^{-\lambda \Delta T}$ therefore

$$\overline{\alpha_n^2} = \overline{\alpha_{n-1}^2} (\text{CORT})^2 + \overline{\zeta^2} \tag{20}$$

The steady state value of $\overline{\alpha_n^2}$ is determined when $n \to \infty$

$$\overline{\alpha_{SS}^2} = (CORT)^2 \overline{\alpha_{SS}^2} + \overline{\zeta^2}$$

or

$$\overline{\alpha_{SS}^2} = \frac{\overline{\zeta^2}}{1 - (CORT)^2} \qquad (21)$$

The steady state mean square drift component is determined as follows:

$$\overline{\alpha_d^2} = \lim_{n \to \infty} (CORT)^2 \overline{\alpha_n^2}$$

$$\overline{\alpha_d^2} = (CORT)^2 \overline{\alpha_{SS}^2} \qquad (22)$$

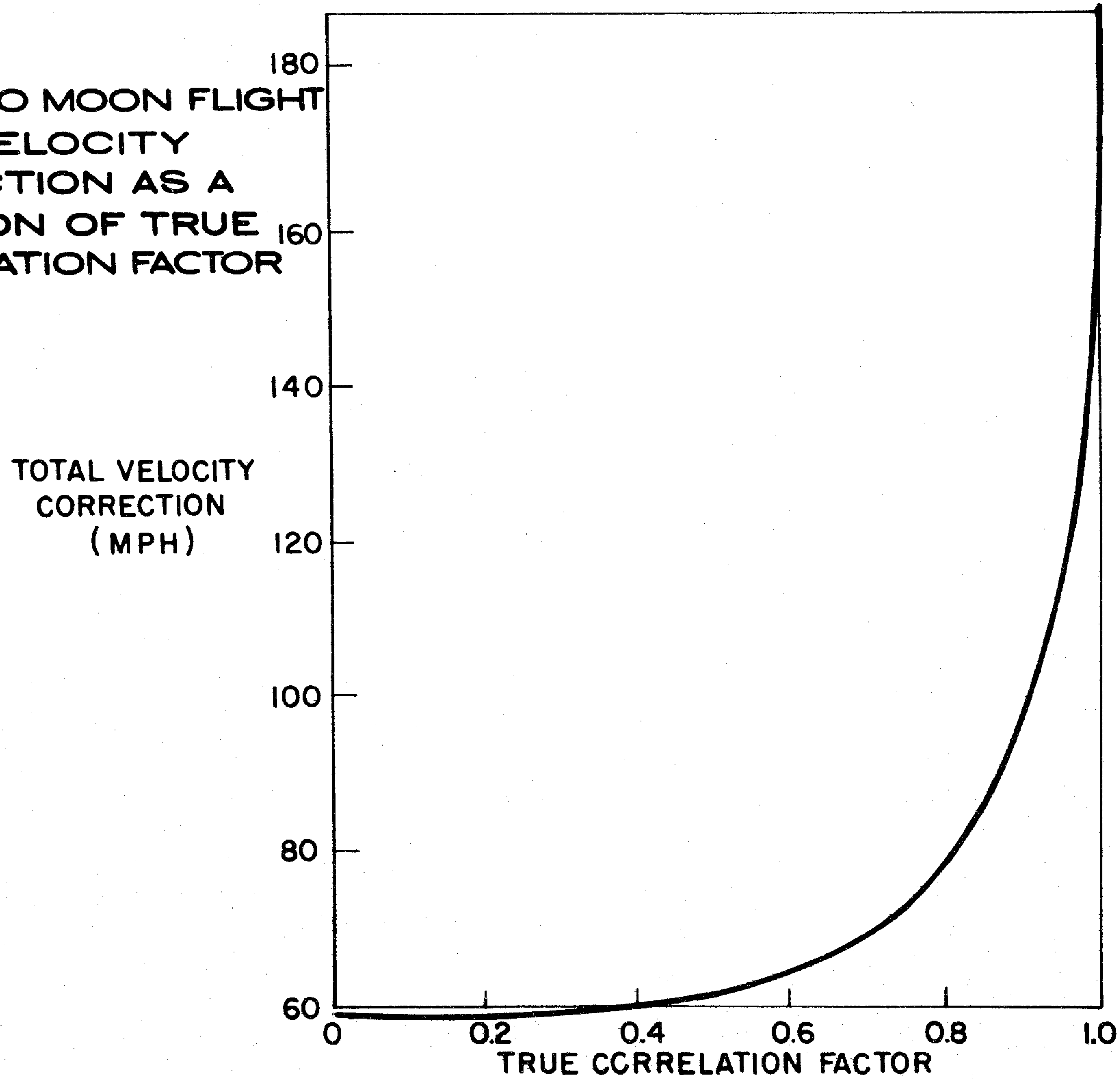
Substitute (21) in (22)

$$\overline{\alpha_d^2} = \frac{\overline{\zeta^2} (CORT)^2}{1 - (CORT)^2} \qquad (23)$$

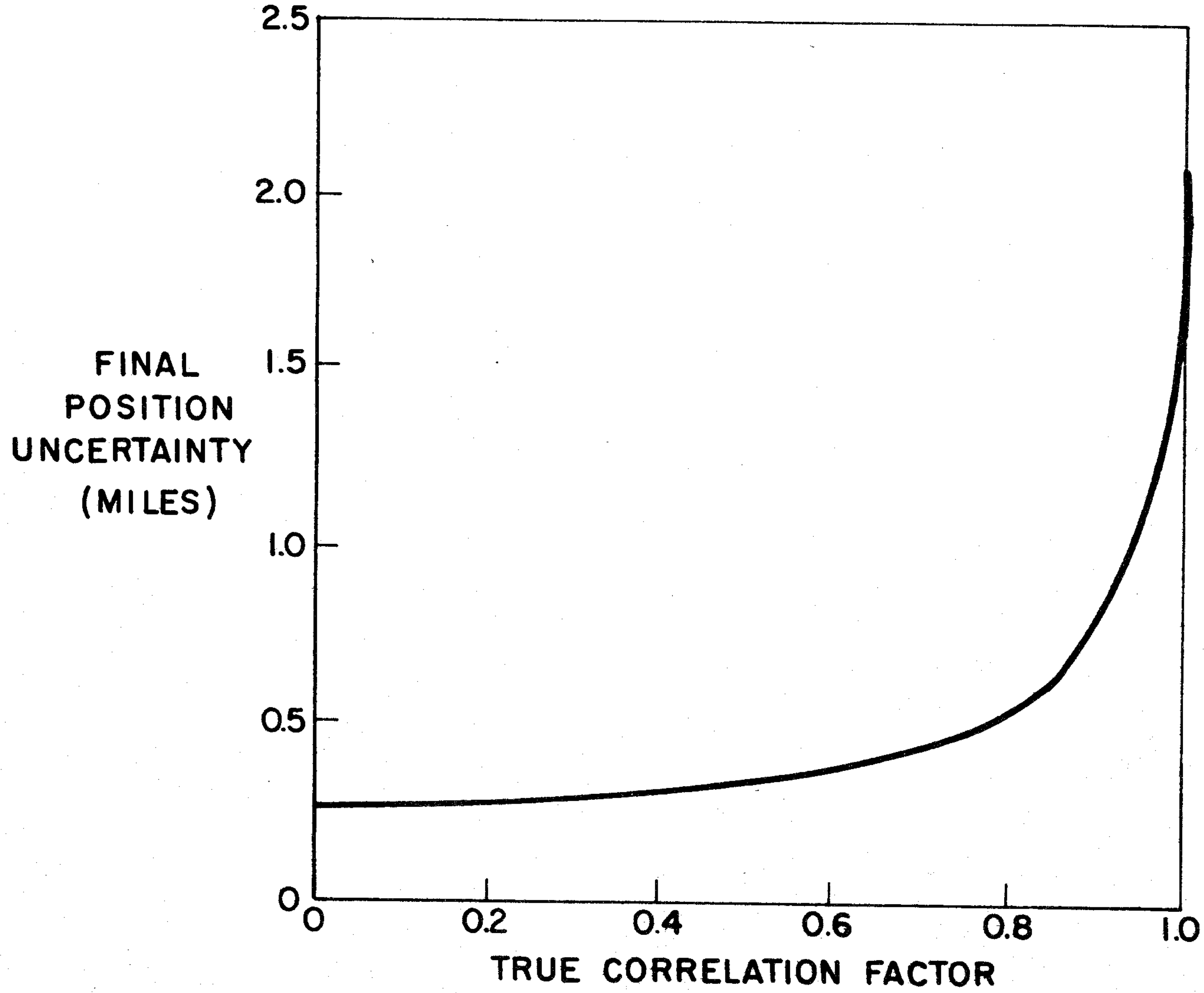
Values of the mean square steady state drift $\overline{\alpha_d^2}$ are assumed and the true correlation factors, CORT, are calculated. Figure 6 presents the true correlation factor as a function of sextant steady state mean square value of the drift component.

In conclusion, the importance of correlated measurement errors is directly related to mean square steady state drift of the sextant. The MIT sextant is within the zero slope region of figures 1 to 5 and therefore correlated measurement errors are not important for the MIT sextant.

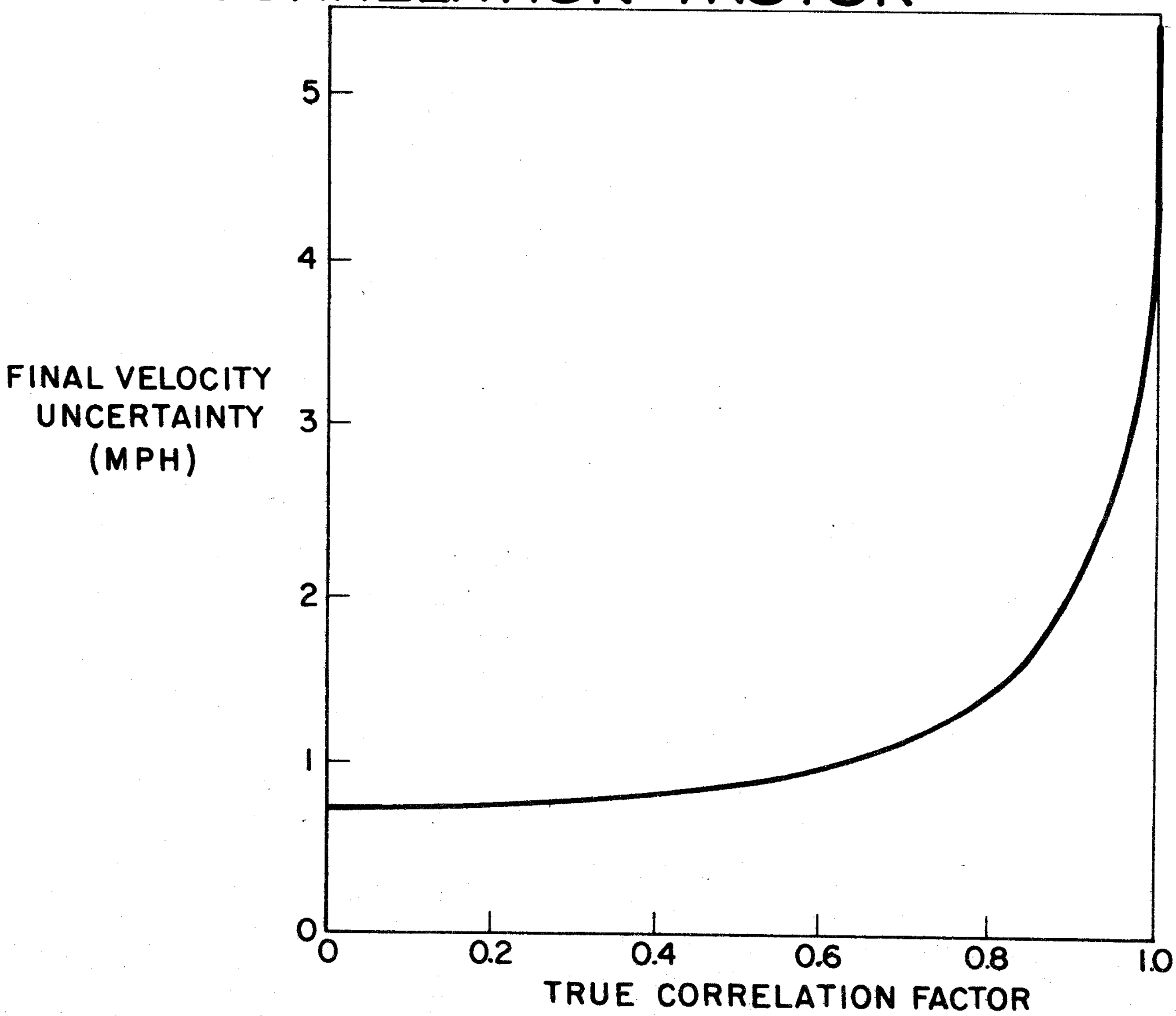
EARTH TO MOON FLIGHT TOTAL VELOCITY CORRECTION AS A FUNCTION OF TRUE CORRELATION FACTOR



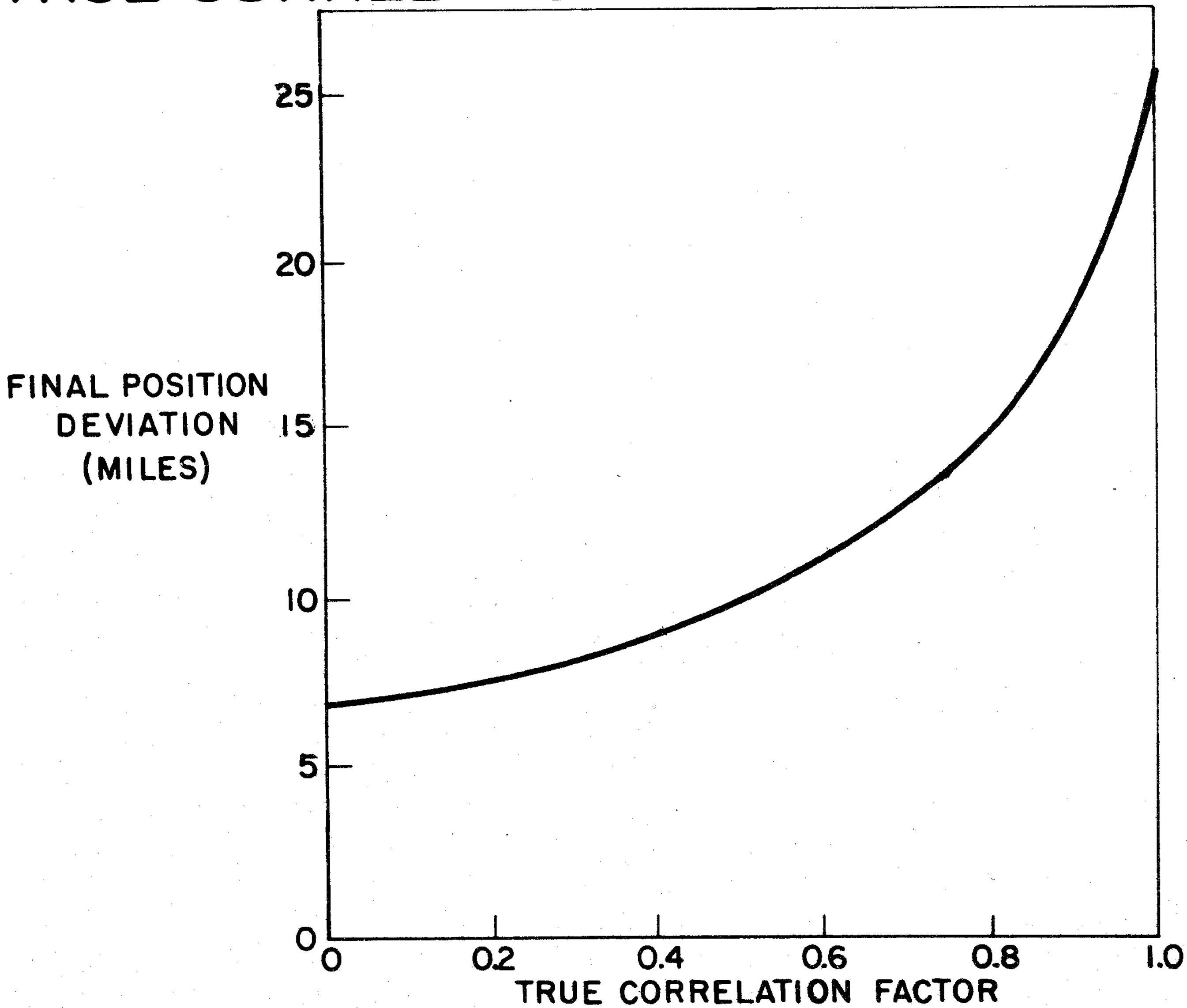
EARTH TO MOON FLIGHT
FINAL POSITION UNCERTAINTY AS A FUNCTION OF TRUE CORRELATION FACTOR



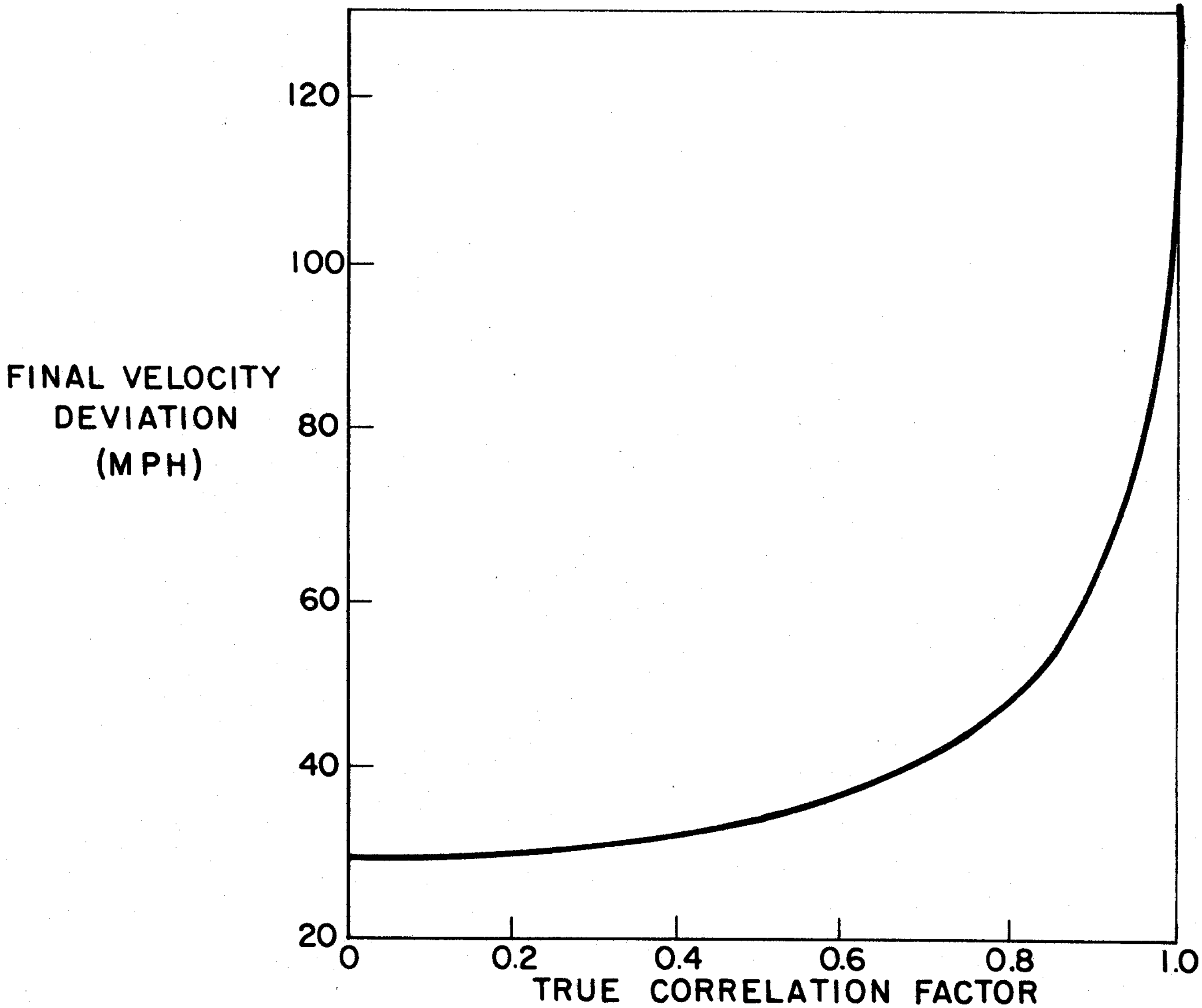
EARTH TO MOON FLIGHT
FINAL VELOCITY UNCERTAINTY AS A FUNCTION OF TRUE CORRELATION FACTOR



EARTH TO MOON FLIGHT
FINAL POSITION DEVIATION AS A FUNCTION OF TRUE CORRELATION FACTOR



EARTH TO MOON FLIGHT
FINAL VELOCITY DEVIATION AS A FUNCTION OF TRUE CORRELATION FACTOR



STEADY STATE MEAN SQUARE SEXTANT DRIFT AS A FUNCTION OF TRUE CORRELATION FACTOR

SEXTANT MODEL $\exp(-\lambda \Delta T)$ = CORRELATION FACTOR = CORT