Massachusetts Institute of Technology
Instrumentation Laboratory
Cambridge, Massachusetts

MEMO

TO: Distribution

FROM: D. S. Baker

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SUBJECT: The Coriolis Equation - A Sequel

A recent memo (Jan. 7 - The Coriolis Equation) has asserted that a contradiction can be arrived at (Eq. 8) commencing with the Coriolis Equation and making use of a certain concept. This memo will attempt to show how this contradiction came about, and that no contradiction truly exists.

The ofttimes called Coriolis Equation as written in Goldstein's "Classical Mechanics" and similarly stated in most other mechanics books is

$$\left(\frac{d\underline{G}}{dt}\right)_S = \left(\frac{d\underline{G}}{dt}\right)_B + \underline{\omega} \times \underline{G} \qquad (4\text{-}100)$$

where $\underline{G}$ is any arbitrary vector, the subscript S denoting the rate of change of $\underline{G}$ as seen by an observer in the space (or fixed) set of axes, the subscript B denoting the rate of change of $\underline{G}$ as observed in the body (or moving) set of axes, and $\underline{\omega}$, the angular velocity of the body set of axes with respect to the space set of axes.

The contradictory equation (Eq. 8) of the Jan. 7 memo is

$$\int_{T_1}^{T_2} \left(\left(\frac{d\underline{G}}{dt}\right)_B + (\underline{\omega} \times \underline{G}) \right) dt = \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt}\right)_B dt$$

where $T_2 > T_1$. How does this non-equality come about? It is stated in the Jan. 7 memo that this non-equality comes about because of the "literal application of two apparently conflicting concepts".

The first concept is that a vector viewed from any coordinate frame is the same physical vector. If one ignores relativistic effects, this concept is not generally contested. The second concept is that the rate of change of a vector as viewed in two coordinate systems is different if the two systems are rotating with respect to each other. This is just a less specific way of stating Eq. (4-100).

Applying the first concept at times T_1 and T_2, $T_2 > T_1$, we have

$$(\underline{G})_{ST_1} = (\underline{G})_{BT_1} \qquad (1)$$

$$(\underline{G})_{ST_2} = (\underline{G})_{BT_2} \qquad (2)$$

as stated in the Jan. 7 memo, the subscripts S and B again denoting the vector as observed in the S or B system, respectively. And the vector at $t = T_2$ as viewed in the S system (non-rotating) can be written as

$$(\underline{G})_{ST_2} = (\underline{G})_{ST_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_S dt \qquad (3)$$

as given by Eq. (4) of the Jan. 7 memo. Next the following statement is made;

$$(\underline{G})_{BT_2} = (\underline{G})_{BT_1} + \int_{T_1}^{T_2} \left(\frac{d\underline{G}}{dt} \right)_B dt. \qquad (4)$$

The contradiction is already evident since substituting for $(\underline{G})_{BT_1}$ and $(\underline{G})_{BT_2}$ from Eqs. (1) and (2), respectively, into Eq. (4) yields a contradiction when compared with Eq. (3).

This memo maintains that Eq. (4) is incorrect. Let us examine a more detailed version of Fig. 1 of the Jan. 7 memo, cf. Fig. 1. For the present, let $T_2 = T_1 + dt$, where dt is a small increment of time. The S system is assumed fixed in the page. The B system is rotating with angular velocity $\underline{\omega}$ about the positive Z axis of either system. One can think of this figure as being a superposition of two pictures taken of the complete system at $t = T_1$, and $t = T_2$. During the time dt, the B system has rotated $d\theta_1 = \omega dt$. Also, during this same dt, the vector $\underline{G}$ has changed by $(d\underline{G})_S$ as seen by the S observer. Under the assumption that the S frame is the fixed or inertial system, $(d\underline{G})_S$ can be thought of as the 'true' change in $\underline{G}$ during dt. If we know $\underline{\omega}$, then $(\underline{\omega} \times \underline{G})$ dt can be calculated, and subtracting it from $(d\underline{G})_S$, one

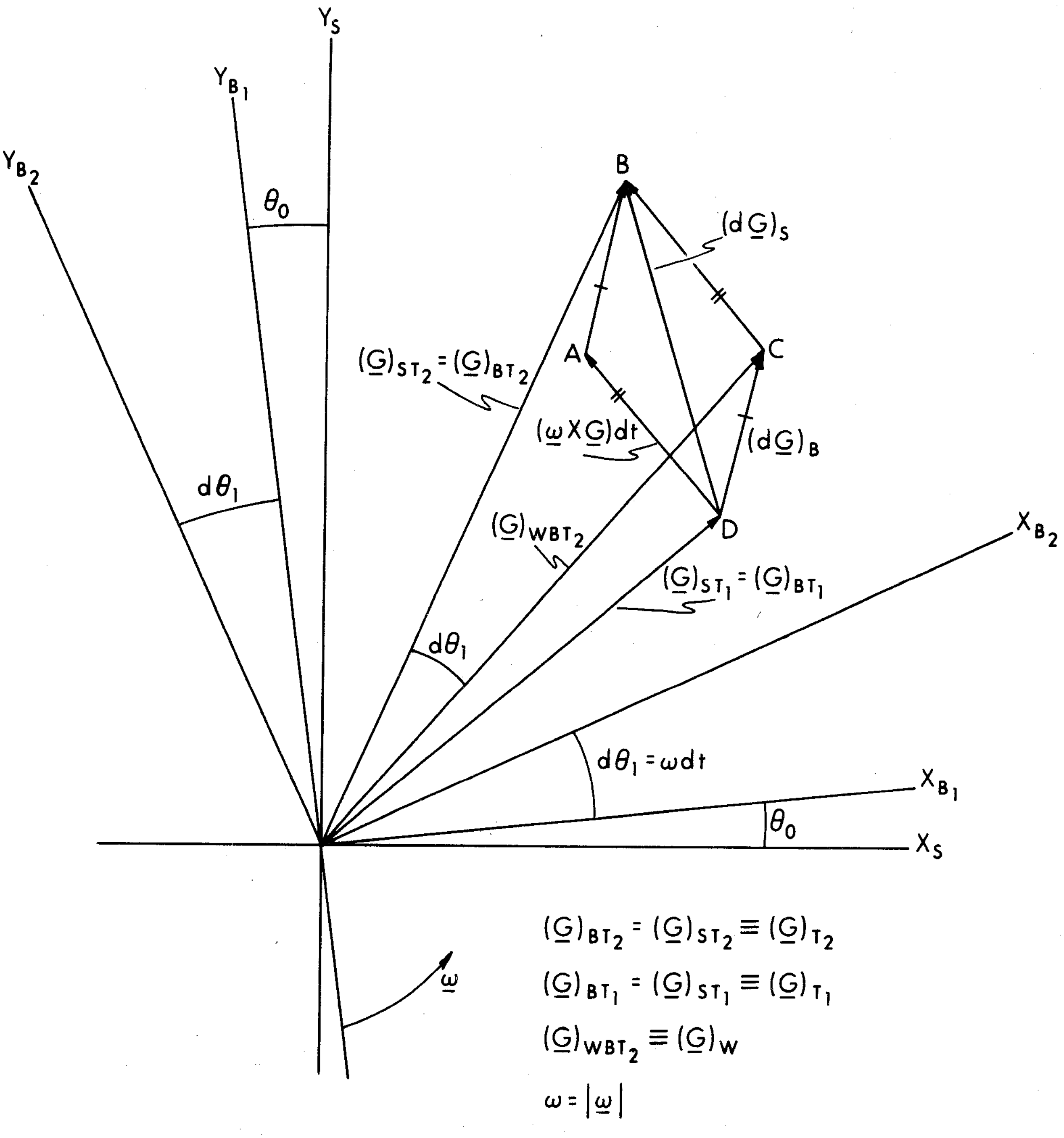

Fig. 1 Vector Diagram of the Coriolis Equation

finds $(d\underline{G})_B$. In other words, the tetrahedren ABCD, or either of the triangles, ABC or BCD, is a vector diagram statement of Eq. (4-100).

From Fig. 1, the equation for $(\underline{G})_{BT_2}$ is seen to be

$$(\underline{G})_{BT_2} = (\underline{G})_{BT_1} + (d\underline{G})_B + (\underline{\omega} \times \underline{G})\, dt$$

$$= (\underline{G})_{ST_1} + (d\underline{G})_B + (\underline{\omega} \times G)\, dt$$

$$= (\underline{G})_{ST_1} + (d\underline{G})_S = (\underline{G})_{ST_2}$$

The vector calculated in Eq. (5) of the Jan. 7 memo is labeled $(\underline{G})_{WBT_2}$ in Fig. 1 and is

$$(\underline{G})_{WBT_2} = (\underline{G})_{T_1} + (d\underline{G})_B \equiv (\underline{G})_W$$

where $(\underline{G})_{T_1}$ is the true vector at $t = T_1$. This 'pseudo' vector, $(\underline{G})_W$, is that which the B observer would mistakenly believe he sees at $t = T_2$ due to the small rotation ωdt. If the B observer is not aware of the rotation $\underline{\omega}\, dt$, he mistakenly thinks he remains in the X_{B_1}, Y_{B_1} frame at $t = T_2$, and believes he is observing $(\underline{G})_W$. The angle between $(\underline{G})_W$ and $(\underline{G})_{T_2}$ is ωdt which is also the angle between the B_2 and B_1 frames. For a unit time, this angle is dependent on ω and not on the angle between the B and S systmes. During subsequent dt's, the B observer makes similar mistakes in the position of $\underline{G}$. To find $\underline{G}_W$ after many time steps, i.e. the vector the B observer believes he is observing, one adds the $(d\underline{G})_B$'s to the original vector $(\underline{G})_{T_1}$.

Although we have spoken of the S frame as fixed and the B frame as moving with respect to it, the reverse choice could have been made (B fixed and S moving). In Fig. 1 the B observer would believe the S frame to be rotating with an angular velocity $\underline{\omega}_1$. The new equation would be

$$\left(\frac{d\underline{G}}{dt}\right)_B = \left(\frac{d\underline{G}}{dt}\right)_S + (\underline{\omega}_1 \times \underline{G}).$$

But, $\underline{\omega}_1 = -\underline{\omega}$; therefore,

$$\left(\frac{d\underline{G}}{dt}\right)_S = \left(\frac{d\underline{G}}{dt}\right)_B + (\underline{\omega} \times \underline{G})$$

which is the original equation. For the same physical situation, or same set of vectors, the designation of one frame as fixed and the other as moving is arbitrary as far as the mechanical dynamics are concerned. If both frames are rotating with respect to a third frame, $\underline{\omega}$ becomes the difference of the rotation rates of B and S with respect to the third frame. Then

$$\left(\frac{d\underline{G}}{dt}\right)_S$$

is not the 'true' rate of change of $\underline{G}$, but only the rate of change as observed in the S system. The point is that Eq. (4-100) is valid for any pair of coordinate systems.

In applications, normally one prefers to call one particular frame the fixed or inertial frame and the other (s) the body or moving frame (s). But this choice is governed by considerations other than pure dynamics. For instance, one might choose his fixed system to be one in which the least Coriolis and centrifugal accelerations are detected.

To this point in the discussion, no comments have been made concerning the components of these vectors in any coordinate system. Equation (4-100) is a vector equation and can be written in component form for calculations in any coordinate system, providing all components are taken in the same system.

Figure 2 is a different version of the same vector diagram of Fig. 1. The components of the differential vectors are projected on the body axes X_{B_1}, Y_{B_1}, and X_{B_2}, Y_{B_2}. Adding the components of $(\underline{\omega} \times \underline{G})\, dt$ and $(d\underline{G})_B$ along the X_{B_1}, Y_{B_1}, axes, one obtains the components of $(d\underline{G})_S$ with respect to the B_1 axes; not the B_2 axes. The B observer has rotated with the body during dt and sees the change $(d\underline{G})_B$ at $t = t_2$ and says the components of it are $(dG)_{BB_2x}$ and $(dG)_{BB_2y}$. The subscript B_2 denotes the frame in which components are being taken. If the body observer is not aware of his rotation,

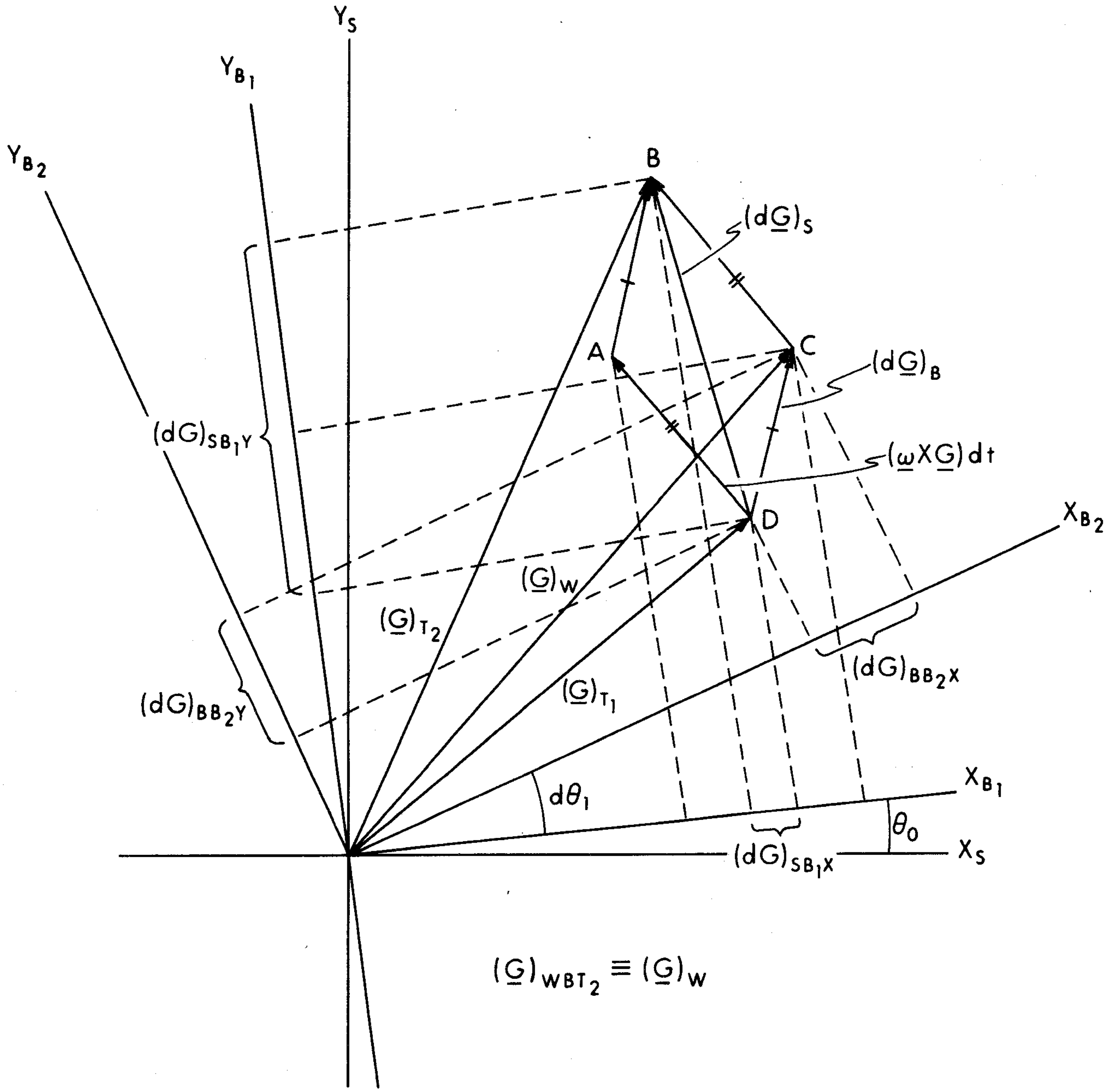

Fig. 2 Components of Equation (4-100) Vectors

he can only calculate a $(d\underline{G})_B$ for each time step; each of which he adds to his previous $\underline{G}_W$ to obtain his present $\underline{G}_W$ in components of his present B system.

If the B observer is aware of his rotation with respect to the S frame, at $t = T_2$ he observes $(d\underline{G})_B$ which he realizes is not the 'correct' change in $\underline{G}$, and that he must add $(\underline{\omega} \times \underline{G})\,dt$ to $(d\underline{G})_B$, both of whose components are in X_{B_1}, Y_{B_1}, to obtain $(d\underline{G})_S$. By adding $(d\underline{G})_S$ to $(\underline{G})_{T_1}$ he has the new vector $\underline{G}_{T_2}$, whose components are still in X_{B_1}, Y_{B_1}. At $t = T_2$ he can observe the components of $\underline{G}_{T_2}$ which are exactly the same components $\underline{G}_W$ has in the B_1 system. Hence, for each time step dt, he uses Eq. (4-100) to calculate the 'correct' change in $\underline{G}$, in the S system, but in components of the B system before the integration step (B_1 system in Fig. 1). If for n integration time steps he stores

$$d\theta_{n-1} = \omega dt, \; \text{----------}, \; d\theta_1 = \omega dt$$

where the $d\theta$'s are the angles thru which the B system has rotated in each time step, the B observer can at any time calculate the present $\underline{G}$ vector in terms of any of the previous B coordinate systems, back to the starting B system X_{B_1}, Y_{B_1}. Nothing yet has been said about the orientation of any of the B systems with respect to the S system.

To use Eq. (4-100) as explained above, no information is needed regarding the orientation of the S system, only that the B system is rotating with angular velocity, $\underline{\omega}$, with respect to it. If the initial orientation (θ_o) of B_1 with respect to the S system is given, and

$$\int_{T_1}^{T_n} \omega dt + \theta_o$$

is stored, then the vector $\underline{G}$ can also be written in the S system. As the Jan. 7 memo points out; in general when making this kind of calculation on a computer, it is more convenient to use a matrix formulation of Eq. (4-100) (Broxmeyer, "Inertial Navigation Systems").

To sum up, both observers, B and S, at any time, t, see the same vector $\underline{G}$; they see different rates of change of $\underline{G}$ as related by Eq. (4-100), and the B observer believes he sees a different vector. If the components of two of the three vectors in (4-100) are known in a particular coordinate system, then the third vector can be calculated in the same system. Hence, it is believed that no contradiction exists either in the concept that a vector is the same vector in all frames of reference, or in Eq. (4-100).